A shock-capturing discontinuous Galerkin method for the numerical solution of inviscid compressible flow

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Outline

Introduction

- **Our aim**: efficient, accurate and robust numerical scheme for the simulation of inviscid and/or viscous compressible flows,
- **Model problem**: system of the Euler equations,
- piecewise regular solution, usually contains discontinuities,
- discontinuous Galerkin method (DGM),
- piecewise polynomial discontinuous approximation,
- shock-capturing scheme $\sim$ system of the Navier-Stokes equations,
- artificial viscosity limiter $\sim$ residual of the entropy equation,
- DGM with IPG variants: nonsymmetric, symmetric and incomplete.

Governing equations (1)

Compressible Euler equations

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{d} \frac{\partial}{\partial x_s} \mathbf{f}_s(\mathbf{w}) = 0 \quad \text{in } \Omega \times (0, T) \qquad (1)$$

- $\mathbf{w} = (\rho, \rho v_1, .., \rho v_d, e)^{\mathrm{T}} \in \boldsymbol{R}^{d+2}$
- Euler fluxes $\mathbf{f}_s : \boldsymbol{R}^{d+2} \to \boldsymbol{R}^{d+2},\ s = 1, \ldots, d$

$$\mathbf{f}_s(\mathbf{w}) = (\rho v_s, \rho v_s v_1 + p\delta_{s1}, .., \rho v_s v_d + p\delta_{sd}, (e+p)\, v_s)^{\mathrm{T}}$$

- properties of fluxes $\mathbf{f}_s$:

$$\begin{aligned} \mathbf{f}_s(\xi \mathbf{w}) &= \xi \mathbf{f}_s(\mathbf{w}),\ \xi \in \boldsymbol{R},\ \xi \neq 0, \\ \mathbf{f}_s(\mathbf{w}) &= \mathbf{A}_s(\mathbf{w})\mathbf{w},\ s = 1, .., d, \\ \mathbf{P} = \sum_{s=1}^{d} \mathbf{A}_s(\mathbf{w}) n_s &= \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1} = \mathbf{P}^{+}(\mathbf{w}, \vec{n}) + \mathbf{P}^{-}(\mathbf{w}, \vec{n}), \end{aligned}$$

Governing equations (2)

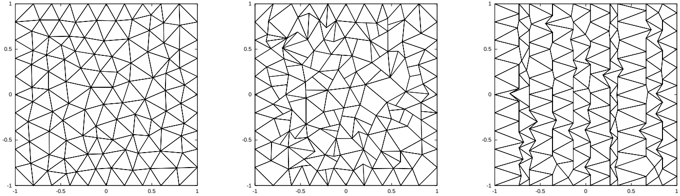
- state equation for perfect gas:

$$p = (\gamma - 1)\left(e - \rho|\mathbf{v}|^2/2\right)$$

- initial condition (IC): $\mathbf{w}(x,0) = \mathbf{w}^0(x)$ in Ω
- boundary conditions (BC):

boundary	character	extrapolated	prescribed
$\partial\Omega_i$ (inlet)	supersonic	—	$\rho, v_1, \ldots, v_d, p$
	subsonic	p	$\rho, v_1, \ldots, v_d$
$\partial\Omega_o$ (outlet)	supersonic	$\rho, v_1, \ldots, v_d, p$	—
	subsonic	$\rho, v_1, \ldots, v_d$	p
$\partial\Omega_w$ (walls)	$\mathbf{v} \cdot \vec{n} = 0$ (impermeability condition)		

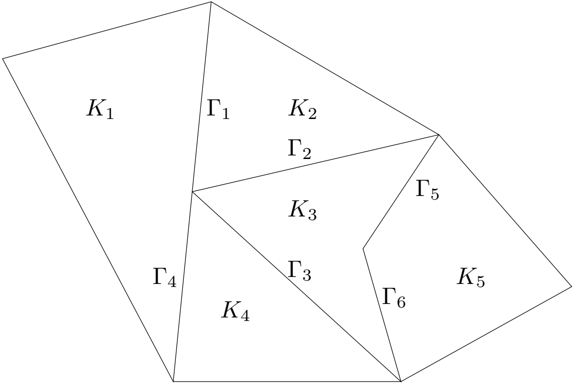
Triangulations



- let $\mathcal{T}_h$, $h > 0$ be a partition of $\overline{\Omega}$
- $\mathcal{T}_h = \{K\}_{K \in \mathcal{T}_h}$, K are polygons/polyhedra
- let $\mathcal{F}_h = \{\Gamma\}_{\Gamma \in \mathcal{F}_h}$ be a set of all edges/faces of $\mathcal{T}_h$,
- we distinguish
  - inner edges/faces $\mathcal{F}_h^I$,
  - 'Dirichlet' edges/faces $\mathcal{F}_h^D$ (i.e., inlet and solid walls),
  - 'Neumann' eges/faces $\mathcal{F}_h^N$ (i.e., outlet),
- we put $\mathcal{F}_h^{ID} \equiv \mathcal{F}_h^I \cup \mathcal{F}_h^D$.

Fictional triangulation

- convex/nonconvex elements with/without hanging nodes



Spaces of discontinuous functions

- let $s_K \geq 1, K \in \mathcal{T}_h$ denote local Sobolev index,
- let $p_K \geq 1, K \in \mathcal{T}_h$ be local polynomial degree,
- we set vectors $\mathbf{s} \equiv \{s_K, K \in \mathcal{T}_h\}$ and $\mathbf{p} \equiv \{p_K, K \in \mathcal{T}_h\}$
- over $\mathcal{T}_h$ we define:
  - *broken Sobolev space*

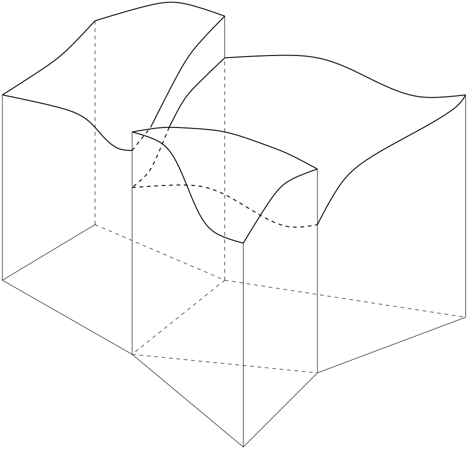
$$H^{\mathbf{s}}(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^{s_K}(K) \; \forall K \in \mathcal{T}_h\}$$

  - the space of piecewise polynomial functions

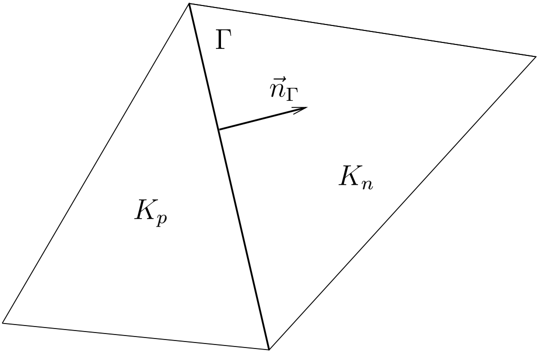
$$S_{hp} \equiv \{v;\ v \in L^2(\Omega),\ v|_K \in P_{p_K}(K) \; \forall K \in \mathcal{T}_h\},$$

- we introduce space $\mathbf{S}_{hp} = \underbrace{S_{hp} \times \ldots \times S_{hp}}_{(d+2) \text{ times}}$

Example of a function from $S_{hp} \subset H^s(\Omega, \mathcal{T}_h)$



Notation - trace, mean value, jump



- $v|_\Gamma^{(p)} \equiv$ trace of $v|_{K_p}$ on Γ and $v|_\Gamma^{(n)} \equiv$ trace of $v|_{K_n}$ on Γ,
- $\langle v \rangle_\Gamma = \frac{1}{2}\left(v|_\Gamma^{(p)} + v|_\Gamma^{(n)} \right)$ and $[v]_\Gamma = v|_\Gamma^{(p)} - v|_\Gamma^{(n)}$,
- $\langle v \rangle_\Gamma \equiv [v]_\Gamma \equiv v|_\Gamma^{(p)}, \ \Gamma \subset \partial\Omega.$

DG formulation

- let $\mathbf{w}$ be a sufficiently regular solution,
- we multiply (1) by $\varphi \in H^2(\Omega, \mathcal{T}_h)^{d+2}$,
- integrate over each $K \in \mathcal{T}_h$,
- apply Green's theorem,
- sum over all $K \in \mathcal{T}_h$,
- we obtain the identity

$$\left(\frac{\partial \mathbf{w}(t)}{\partial t}, \varphi\right) + \widetilde{\mathbf{b}}_h(\mathbf{w}(t), \varphi) = 0 \quad \forall\, \varphi \in H^2(\Omega, \mathcal{T}_h)^{d+2},\ \forall\, t \in (0, T) \tag{2}$$

Inviscid terms

$$\widetilde{\mathbf{b}}_h(\mathbf{w},\varphi) = \sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma \mathbf{H}\left(\mathbf{w}|_\Gamma^{(p)}, \mathbf{w}|_\Gamma^{(n)}, \vec{n}_\Gamma\right)\cdot[\varphi]_\Gamma \,\mathrm{d}S - \sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^{d} \mathbf{A}_s(\mathbf{w})\mathbf{w}\cdot\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x,$$

where **H** is the Vijayasundaram numerical flux,

$$H\left(\mathbf{w}|_\Gamma^{(p)}, \mathbf{w}|_\Gamma^{(n)}, \vec{n}_\Gamma\right) = \mathbf{P}^{+}\left(\langle\mathbf{w}_h\rangle,\vec{n}\right)\mathbf{w}_h|_\Gamma^{(p)} + \mathbf{P}^{-}\left(\langle\mathbf{w}_h\rangle,\vec{n}\right)\mathbf{w}_h|_\Gamma^{(n)}$$

Space semi-discrete problem

- method of lines for the Euler equations,
- $S_{hp} \subset H^2(\Omega, \mathcal{T}_h) \Rightarrow$ identity (2) makes sense for $\mathbf{w}_h, \varphi_h \in \mathbf{S}_{hp}$,
- approximate solution $\mathbf{w}_h(t) \in \mathbf{S}_{hp}$ satisfies the identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{w}_h(t), \varphi_h) + \widetilde{\mathbf{b}}_h(\mathbf{w}_h(t), \varphi_h) = 0 \qquad \forall \varphi_h \in \mathbf{S}_{hp},\ t \in (0, T), \tag{3}$$

with $\mathbf{w}_h(0)$ satisfying IC,

- semi-discrete problem (3) represents ODEs,
- explicit method $\Rightarrow$ high restriction on time step,
- full implicit method $\Rightarrow$ system of nonlinear equations

Semi-implicit method

BDF scheme + linearization of (3)

Linearization of the inviscid fluxes

- inviscid terms: for $\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h \in \mathbf{S}_{hp}$

$$\mathbf{b}_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) = -\sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{d} \mathbf{A}_s(\tilde{\mathbf{w}}_h)\mathbf{w}_h \cdot \frac{\partial \varphi_h}{\partial x_s} \, \mathrm{d}\mathbf{x}$$

$$+ \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \left(\mathbf{P}^+ \left(\langle \tilde{\mathbf{w}}_h \rangle, \vec{n} \right) \mathbf{w}_h|_\Gamma^{(p)} + \mathbf{P}^- \left(\langle \tilde{\mathbf{w}}_h \rangle, \vec{n} \right) \mathbf{w}_h|_\Gamma^{(n)} \right) \cdot [\varphi_h] \mathrm{d}S,$$

- linear with respect to $\mathbf{w}_h, \varphi_h$,
- consistent with $\widetilde{\mathbf{b}}_h$:

$$\mathbf{b}_h(\mathbf{w}_h, \mathbf{w}_h, \varphi_h) = \widetilde{\mathbf{b}}_h(\mathbf{w}_h, \varphi_h) \;\; \forall \, \mathbf{w}_h, \varphi_h \in \mathbf{S}_{hp}.$$

Higher order semi-implicit BDF-DGM scheme

- partition of $(0, T) \Rightarrow t_0 < t_1 < \cdots < t_r$, $\tau_k \equiv t_{k+1} - t_k$,

General higher order scheme

$$\frac{1}{\tau_k}\left(\sum_{l=0}^{n} \alpha_l \mathbf{w}_h^{k+1-l}, \varphi_h\right) + \mathbf{b}_h\left(\sum_{l=1}^{n} \beta_l \mathbf{w}_h^{k+1-l}, \mathbf{w}_h^{k+1}, \varphi_h\right) = 0$$
$$\forall \varphi_h \in \mathbf{S}_{hp}, \ k = n-1, \dots, r-1,$$

- α_l, β_l - coefficients of BDF,
- $\mathbf{w}_h^0$ is $\mathbf{S}_{hp}$-approximation of $\mathbf{w}^0$,
- $\mathbf{w}_h^l, 1 \leq l \leq n-1$ given by a one-step method.

Drawback

High order approximation of discontinuous solution $\Rightarrow$ Gibbs-type oscillations $\Rightarrow$ instability.

Artificial viscosity limiter (1)

Main idea: add artificial term to (1) in the form corresponds to the viscous part of the system of the Navier-Stokes equations but with the variable Reynolds number

$$\left.\frac{1}{Re}\right|_K \approx \mu_{art}(\mathbf{w}_h, K) \Rightarrow \delta_S \sim \mathcal{O}(\mu_{art})$$

- new system $\sim$ compressible Navier-Stokes equations

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{d} \frac{\partial}{\partial x_s} \mathbf{f}_s(\mathbf{w}) = \mu_{art} \sum_{s=1}^{d} \frac{\partial}{\partial x_s} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) \quad \text{in } (0, T) \times \Omega$$

- viscous fluxes (without Re)

$$\mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) = \left(0, \tau_{1s}, .., \tau_{ds}, \sum_{r=1}^{d} \tau_{rs} v_r + \frac{\gamma}{Pr} \frac{\partial \theta}{\partial x_s}\right)^{\mathrm{T}}, \quad s = 1, .., d$$

Artificial viscosity limiter (2)

- viscous part of the stress tensor (without *Re*)

$$\tau_{rs} = \left[\left(\frac{\partial v_s}{\partial x_r} + \frac{\partial v_r}{\partial x_s} \right) - \frac{2}{3} \mathrm{div}(\mathbf{v}) \delta_{rs} \right], \;\; r, s = 1, .., d,$$

- properties of viscous fluxes $\mathbf{R}_s$:

$$\mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}) = \sum_{k=1}^{d} \mathbf{K}_{s,k}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k}, \quad s = 1, .., d,$$

where $\mathbf{K}_{s,k} \in R^{(d+2)\times(d+2)}, \quad s, k = 1, .., d,$

- add equation of total energy: $e = c_V \rho \theta + \rho |\mathbf{v}|^2/2,$

approach ≡ solution of N.S. equations with "do nothing" BC

Entropy-based vanishing viscosity (1)

- nonlinear viscosity $\approx$ residual of the entropy equation $S_{res}(\mathbf{w})$ [Guermond, Pasquetti, 08]
- entropy: $S = \frac{1}{\gamma - 1} \ln\left(\frac{p}{\rho^\gamma}\right)$ (γ - Poisson adiabatic constant)
- entropy form of the energy equation:

$$\frac{\partial \rho S}{\partial t} + \operatorname{div}(\rho S \mathbf{v}) = \frac{D(\mathbf{v})}{\theta} + \mu_{art} \frac{\gamma}{Pr} \frac{\operatorname{div}(\nabla \theta)}{\theta}$$

where $D(\mathbf{v})$ is dissipation

$$D(\mathbf{v}) = -\frac{2}{3}\mu_{art}(\operatorname{div}(\mathbf{v}))^2 + 2\mu_{art}\mathbb{D}(\mathbf{v}) \cdot \mathbb{D}(\mathbf{v})$$

with $\mathbb{D}(\mathbf{v})$ as deformation velocity tensor

Entropy-based vanishing viscosity (2)

- discrete entropy residual (weak formulation):

$$\int_\Omega S_{res}(\mathbf{w})\varphi_h \,\mathrm{d}x = \int_\Omega \left(\frac{\partial \rho S}{\partial t} + \mathrm{div}(\rho S \mathbf{v}) - \frac{D(\mathbf{v})}{\theta} - \mu_{art} \frac{\gamma}{Pr} \frac{\mathrm{div}(\nabla \theta)}{\theta} \right) \varphi_h \,\mathrm{d}x$$

- $\mathrm{supp}(\varphi_h) \subset K$ + Green's theorem imply

$$\begin{aligned} \int_K S_{res}(\mathbf{w})|_K \varphi_h = & \int_K \frac{\partial \rho S}{\partial t} \varphi_h \,\mathrm{d}x + \int_{\partial K} \rho S (\mathbf{v} \cdot \vec{n}) \varphi_h \,\mathrm{d}S - \int_K \rho S \mathbf{v} \cdot \nabla \varphi_h \,\mathrm{d}x \\ & - \int_K \frac{D(\mathbf{v})}{\theta} \varphi_h \,\mathrm{d}x - \frac{\gamma}{Pr} \int_{\partial K} \mu_{art}(K) \nabla \theta \cdot \vec{n} \frac{\varphi_h}{\theta} \,\mathrm{d}S \\ & + \frac{\gamma}{Pr} \int_K \mu_{art}(K) \nabla \theta \cdot \nabla \left(\frac{\varphi_h}{\theta} \right) \,\mathrm{d}x \quad \forall \varphi_h \in P_{p_K} \end{aligned}$$

- $S_{res}(\mathbf{w})$ is L^2-projection onto $\mathcal{S}_{hp}$, i.e. $S_{res}(\mathbf{w})|_K \in P_{p_K}$

Entropy-based vanishing viscosity (3)

- maximal values (to limit viscosity)

$$\mu_{max} = \nu_{max} \max_{K\in\mathcal{T}_h} \rho \quad \text{and} \quad \nu_{max} = \frac{h_K}{p_K} \max_{K\in\mathcal{T}_h} \left(|\mathbf{v}| + \sqrt{\gamma\theta} \right)$$

- finally we set

$$\mu_{art}(x)|_K = \min(\mu_{max}, \alpha L h_K \cdot |S_{res}(x)|)$$

with h_K...element size, L...characteristic length

- regions with smooth solution $\mathbf{w} \Rightarrow S_{res}(\mathbf{w}) \approx 0$
- regions with shocks $\Rightarrow S_{res}(\mathbf{w})$ is large
- simplification: Laplacian artificial viscosity

$$\mathbf{K}_{s,k} = \delta_{sk}\mathbf{I} \Rightarrow \sum_{s=1}^{d} \frac{\partial}{\partial x_s} \mathbf{R}_s(\mathbf{w}, \nabla\mathbf{w}) = \Delta\mathbf{w}$$

- simplification: $\varphi_h|_K = \chi_K \Rightarrow \mu_{art}$ piecewise constant

Artificial viscous and penalty terms

$$\begin{aligned}
\tilde{\mathbf{a}}_h(\mathbf{w},\varphi) &= \sum_{K\in\mathcal{T}_h}\int_K \mu_{art}(\mathbf{w})\left(\sum_{k=1}^{d}\mathbf{K}_{s,k}(\mathbf{w})\frac{\partial \mathbf{w}}{\partial x_k}\right)\cdot\nabla\varphi\,\mathrm{d}x \\
&- \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sum_{s=1}\left\langle \mu_{art}(\mathbf{w})\left(\sum_{k=1}^{d}\mathbf{K}_{s,k}(\mathbf{w})\frac{\partial \mathbf{w}}{\partial x_k}\right)\right\rangle n_s\cdot[\varphi]\,\mathrm{d}S \\
&- \Theta\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sum_{s=1}\left\langle \mu_{art}(\mathbf{w})\sum_{k=1}^{d}\mathbf{K}^{\mathrm{T}}_{k,s}(\mathbf{w})\frac{\partial \varphi}{\partial x_k}\right\rangle n_s\cdot[\mathbf{w}]\,\mathrm{d}S,
\end{aligned}$$

where $\Theta = 1\ (\text{SIPG}), 0\ (\text{IIPG}), -1\ (\text{NIPG})$

$$\mathbf{J}_h^\sigma(\mathbf{w},\varphi) = \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma[\mathbf{w}]\cdot[\varphi]\,\mathrm{d}S \quad \text{with} \quad \sigma|_\Gamma \equiv \mu_{art}\frac{C_W}{d(\Gamma)},\ C_W>0,$$

$$d(\Gamma) = \min(h_{K_p}/p_{K_p}, h_{K_n}/p_{K_n}),\ \Gamma = K_p\cap K_n$$

Linearization of viscous terms

$$
\begin{aligned}
\mathbf{a}_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) \quad &= \sum_{K \in \mathcal{T}_h} \int_K \mu_{art} \left(\sum_{k=1}^{d} \mathbf{K}_{s,k}(\tilde{\mathbf{w}}_h) \frac{\partial \mathbf{w}_h}{\partial x_k} \right) \cdot \nabla \varphi_h \, \mathrm{d}x \\
&- \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \sum_{s=1} \left\langle \mu_{art} \left(\sum_{k=1}^{d} \mathbf{K}_{s,k}(\tilde{\mathbf{w}}_h) \frac{\partial \mathbf{w}_h}{\partial x_k} \right) \right\rangle n_s \cdot [\varphi_h] \, \mathrm{d}S \\
&- \Theta \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \sum_{s=1} \left\langle \mu_{art} \sum_{k=1}^{d} \mathbf{K}_{k,s}^{\mathrm{T}}(\tilde{\mathbf{w}}_h) \frac{\partial \varphi_h}{\partial x_k} \right\rangle n_s \cdot [\mathbf{w}_h] \, \mathrm{d}S
\end{aligned}
$$

- linear with respect to $\mathbf{w}_h, \varphi_h$,
- consistency:

$$\mathbf{a}_h(\mathbf{w}_h, \mathbf{w}_h, \varphi_h) = \tilde{\mathbf{a}}_h(\mathbf{w}_h, \varphi_h) \;\; \forall\, \mathbf{w}_h, \varphi_h \in \mathbf{S}_{hp}$$

.

Discrete problem

- we set

$$\mathbf{c}_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) := \mathbf{a}_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) + \mathbf{b}_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) + \mathbf{J}_h(\mathbf{w}_h, \varphi_h),$$

- partition $(0, T) \to t_0 < t_1 < \cdots < t_r$, $\tau_k \equiv t_{k+1} - t_k$,

Shock-capturing scheme

$$\frac{1}{\tau_k}\left(\sum_{l=0}^{n} \alpha_l \mathbf{w}_h^{k+1-l}, \varphi_h\right) + \mathbf{c}_h\left(\sum_{l=1}^{n} \beta_l \mathbf{w}_h^{k+1-l}, \mathbf{w}_h^{k+1}, \varphi_h\right) = 0$$

$$\forall \varphi_h \in \mathbf{S}_{hp}, \ \ k = n-1, \dots, r-1,$$

- α_l, β_l - coefficients of BDF,
- $\mathbf{w}_h^0$ is $\mathbf{S}_{hp}$-approximation of $\mathbf{w}^0$,
- $\mathbf{w}_h^l, 1 \le l \le n-1$ given by a one-step method.

Basis $\mathbf{S}_{hp}$

- local basis:
  $B_i = \left\{\psi_{ij},\ \psi_{ij} \in \mathbf{S}_{hp},\ \text{supp}(\psi_{ij}) \subset K_i,\ j = 1, .., dof_i\right\},$
  $i = 1, .., \#\mathcal{T}_h,$
- global basis:
  $B = \left\{\psi_{ij},\ \psi_{ij} \in B_i,\ j = 1, .., dof_i,\ i = 1, .., \#\mathcal{T}_h\right\},$

Linear algebraic representation

$$\mathbf{w}_h^k(x) = \sum_{K_i \in \mathcal{T}_h} \sum_{j=1}^{dof_i} \xi_{kij}\psi_{ij},\ x \in \Omega,\ k = 0, 1, .., r,$$

$$\mathbf{w}_h^k(x) \leftrightarrow \mathbf{W}_k = \left\{\xi_{kij}\right\}_{ij} \in \boldsymbol{R}^{DOF},\ DOF = \sum_{K_i \in \mathcal{T}_h} dof_i.$$

Linear algebraic problem

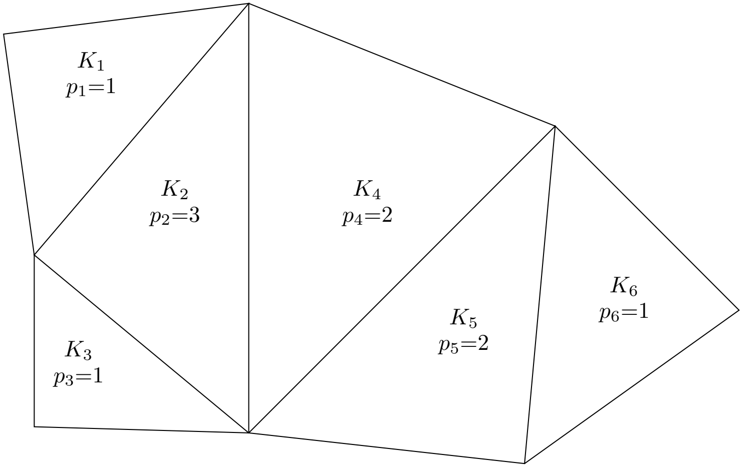
BDF-DG scheme

$$(\mathbf{M} + \tau_k \mathbf{C}_k(\mathbf{W}_k)) = \mathbf{q}_k,\ k = 1, .., r,$$

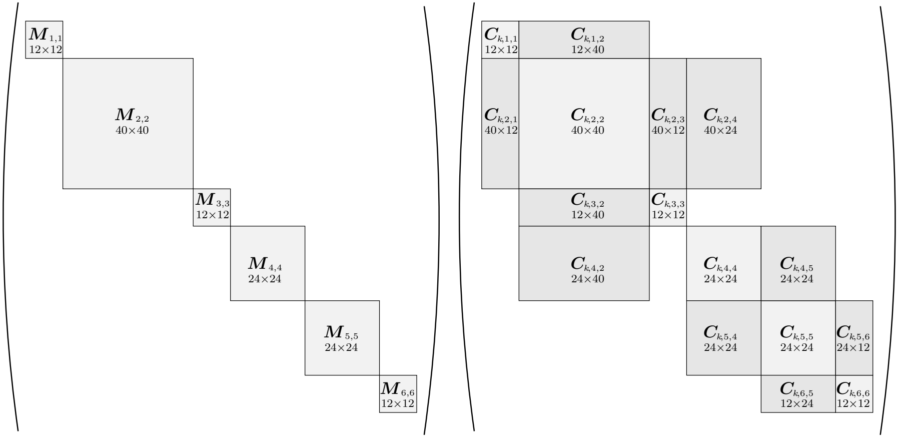
where

- $\mathbf{W}_k$ unknown vector,
- $\mathbf{M}$ mass matrix,
- $\mathbf{C}_k$ "flux" matrix representing inviscid, viscous and penalty terms,
- $\mathbf{q}_k$ right-hand side (BC),
- τ_k time step.

Fictional triangulation



Matrix structure of $\mathbf{M}$ and $\mathbf{C}_k$



Other implementation aspects

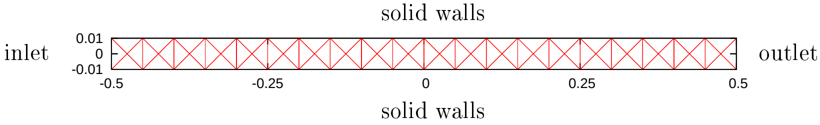
- restarted GMRES with block diagonal preconditioning,
- orthonormal basis of $S_{hp} \Rightarrow \mathbf{M} =$ identity matrix,
- time step for semi-implicit scheme: adaptive and/or heuristic choice,
- penalty parameter σ is set according to NIPG variant: $C_W = 1.0$,
- 2D Fortran code: program ADGFEM [Dolejší] (KNM MFF UK)

Lax problem (1)

- 1D test case: shock, contact discontinuity, rarefaction wave
- computational domain $[-0.5, 0.5]$,
- initial conditions:

$$(\rho, v_1, p) = \begin{cases} (0.445, \quad 0.698, \quad 3.528) & \text{if } x \leq 0, \\ (0.5, \quad 0.0, \quad 0.571) & \text{if } x > 0. \end{cases}$$

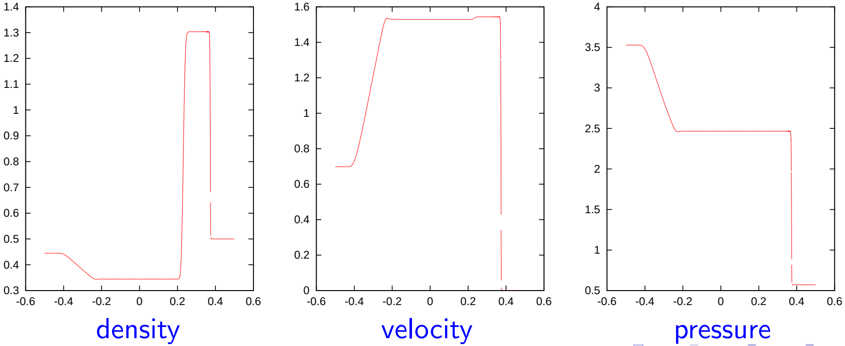
- reflecting boundary conditions,
- interval $[-0.5, 0.5]$ replaced with 2D computational domain



- added BC on solid walls: $\mathbf{v} \cdot \vec{n} = 0 \Leftrightarrow v_2 = 0$

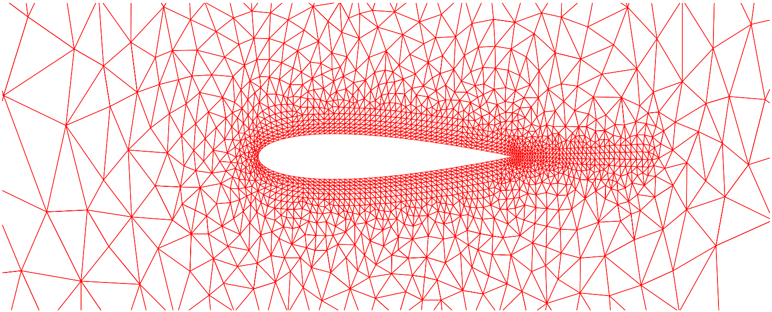
Lax problem (2)

- P_3 approximation, BDF 1^{st} order
- uniform grid with 2 000 elements, final time $T = 0.15$,
- distribution of density, velocity and pressure



NACA 0012 profile - steady flow (1)

- steady transonic inviscid flow around the NACA0012 airfoil
- $M = 0.8$, $\alpha = 1.25^\circ$ (non-symmetric flow),
- triangular adaptive refined grid having 4 544 elements,
- adaptive BDF scheme, $P_1 - P_3$ approximation



NACA 0012 profile - steady flow (2)

- test case from project ADIGMA:
  A EUROPEAN PROJECT ON THE DEVELOPMENT OF ADAPTIVE HIGHER ORDER VARIATIONAL METHODS FOR AEROSPACE APPLICATIONS
- coefficients of drag c_D and lift c_L (comparison with UNST)

method	c_D	c_L	DOF
BDF-DGM – P_1	0.02426	0.33684	54 528
BDF-DGM – P_2	0.02300	0.34065	109 056
BDF-DGM – P_3	0.02277	0.35587	181 760
NSDG2D – P_3 [UNST]	0.02254	0.36448	181 760
NSDG2D – P_5 [UNST]	0.02276	0.35366	381 696

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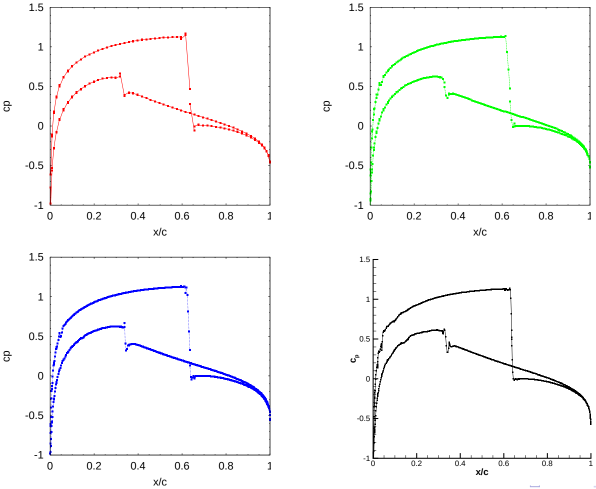
NACA 0012 profile - steady flow (2)

- test case from project ADIGMA:
  A EUROPEAN PROJECT ON THE DEVELOPMENT OF ADAPTIVE HIGHER ORDER VARIATIONAL METHODS FOR AEROSPACE APPLICATIONS
- coefficients of drag c_D and lift c_L (comparison with UNST)

method	c_D	c_L	DOF
BDF-DGM – P_1	0.02426	0.33684	54 528
BDF-DGM – P_2	0.02300	0.34065	109 056
BDF-DGM – P_3	0.02277	0.35587	181 760
NSDG2D – P_3 [UNST]	0.02254	0.36448	181 760
NSDG2D – P_5 [UNST]	0.02276	0.35366	381 696

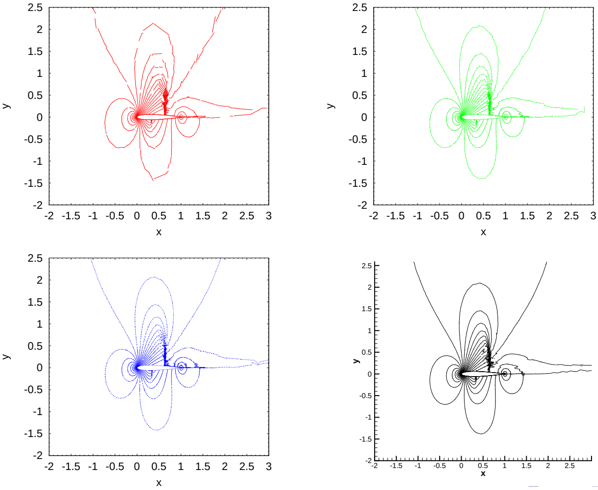
NACA 0012 profile - pressure coefficient

- comparison of P_1, P_2, P_3 and P_5 (UNST) approximations

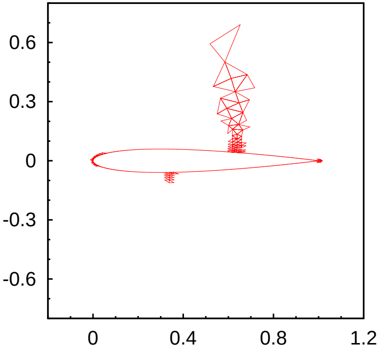


NACA 0012 profile - Mach number isolines

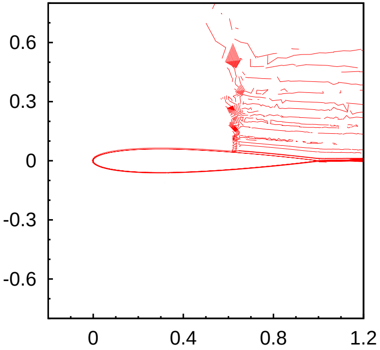
- comparison of P_1, P_2, P_3 and P_5 (UNST) approximations



NACA 0012 profile - artificial viscosity vs. entropy

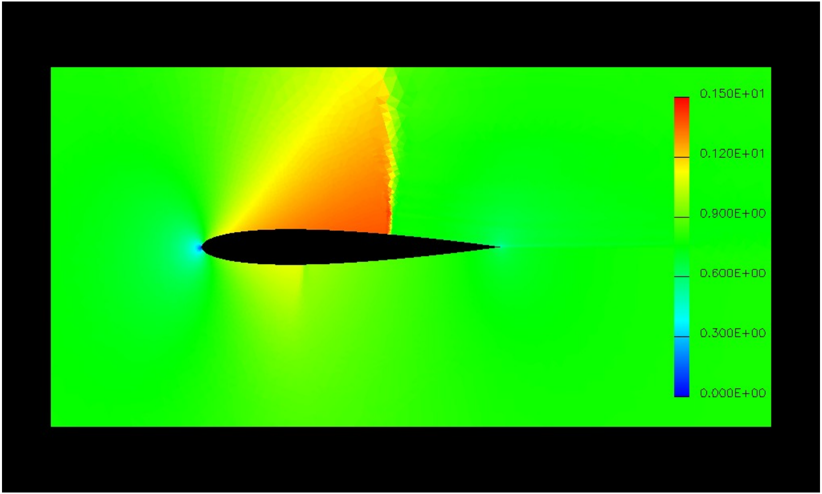


$\mu_{art} > 10^{-3}$

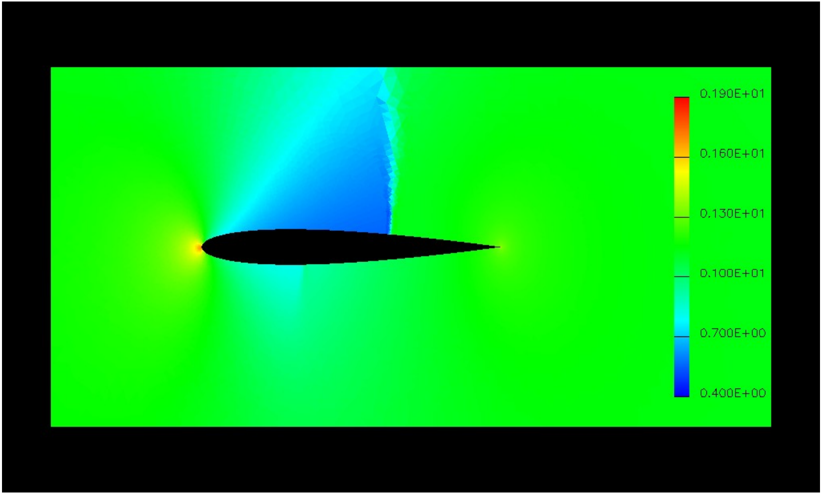


entropy

NACA 0012 - Mach number distribution, $t \to \infty$

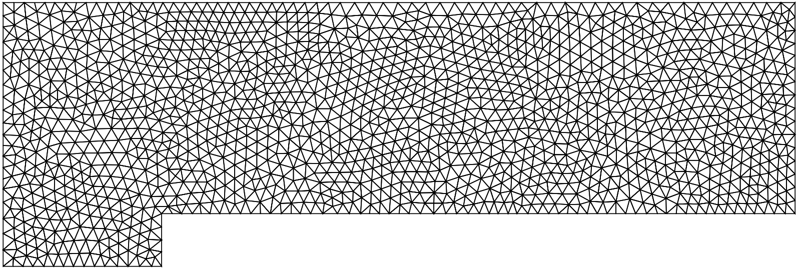


NACA 0012 - pressure distribution, $t \to \infty$

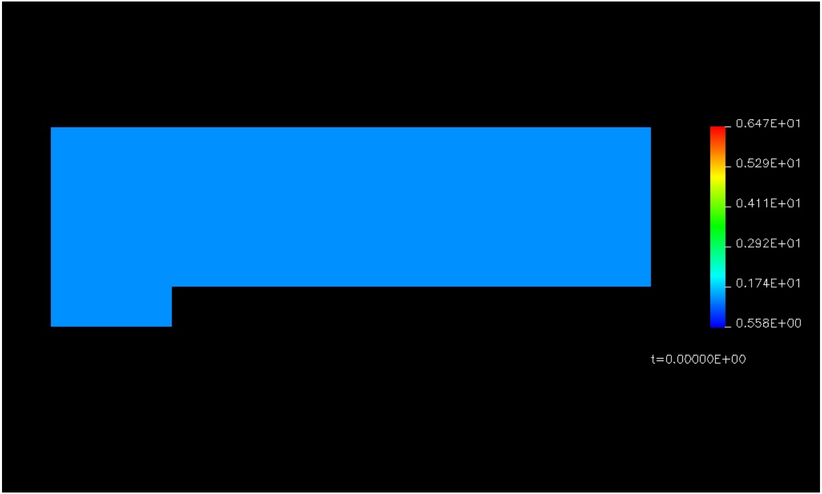


Forward facing step - unsteady flow

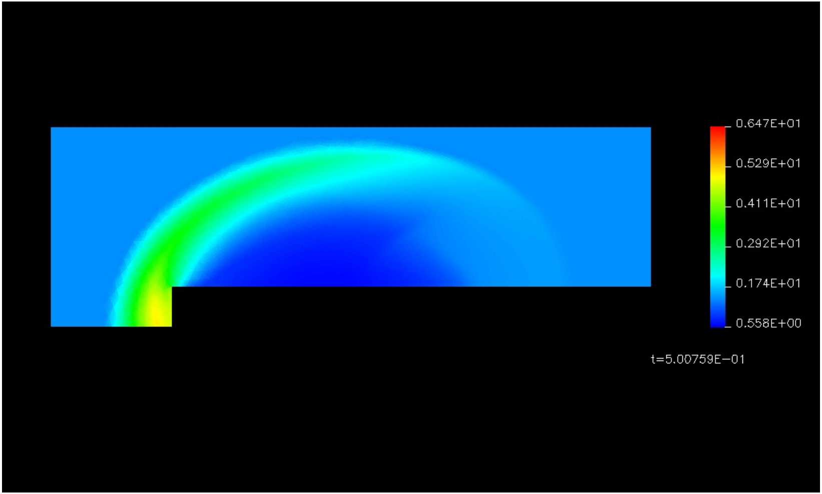
- benchmark of unsteady inviscid flow
- IC: $\mathbf{w} = (1.4,\ 3,\ 0,\ 1)$, inflow/ouflow BC form IC,
- simulation for $t \in (0, 4)$,
- grid with 2 914 triangles, P_2 approximation, 2^{nd} order in time



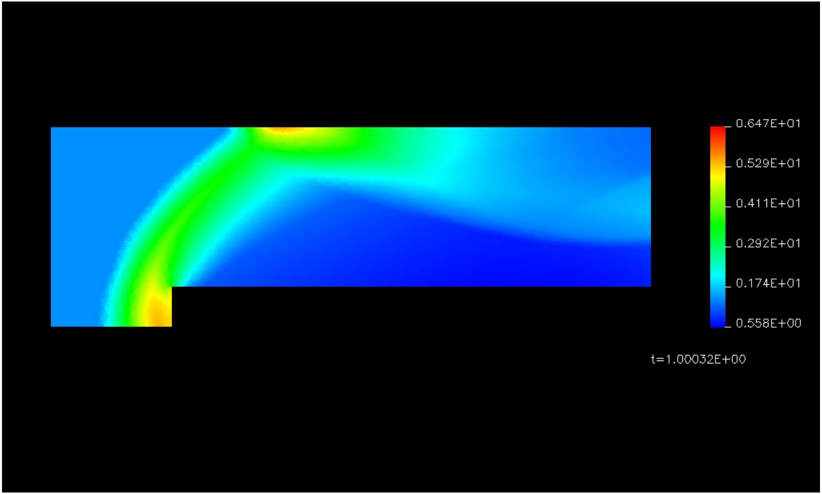
FFS - density distribution, t=0.0



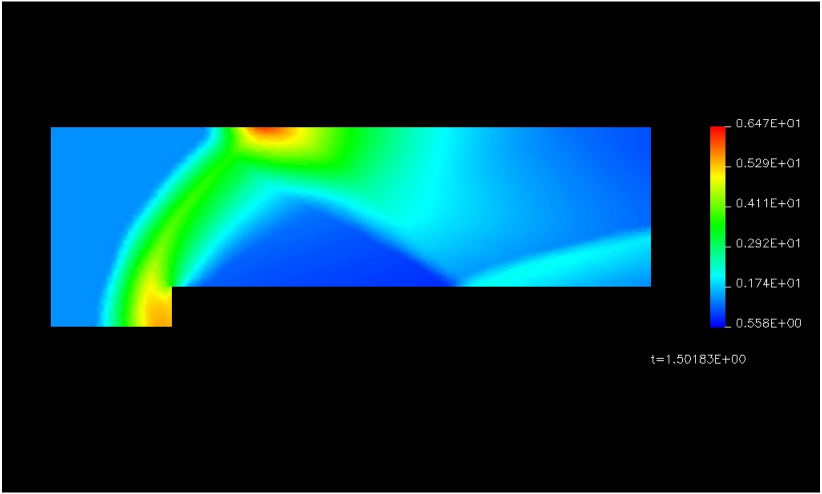
FFS - density distribution, t=0.5



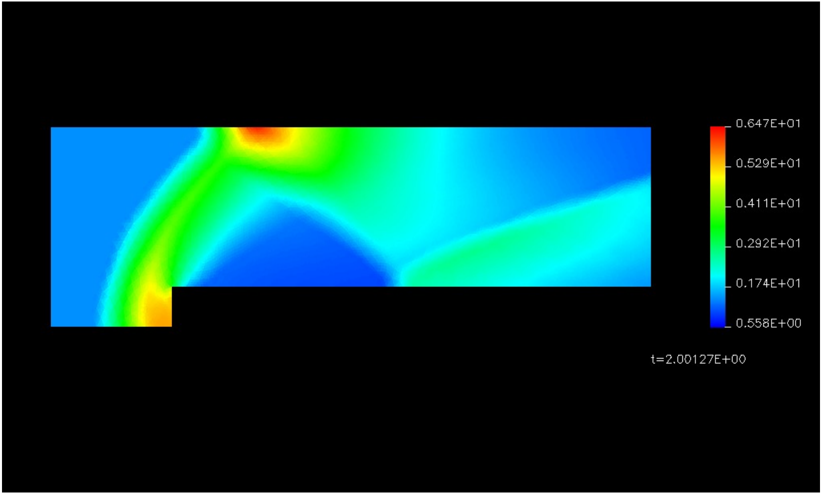
FFS - density distribution, t=1.0



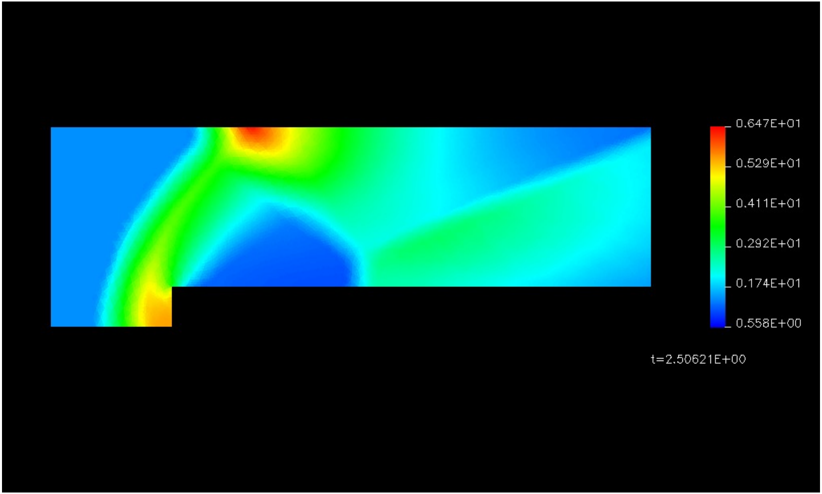
FFS - density distribution, t=1.5



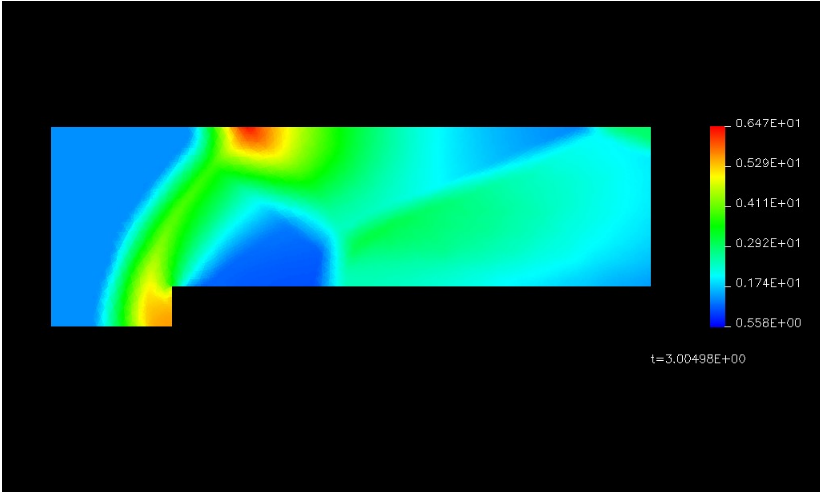
FFS - density distribution, t=2.0



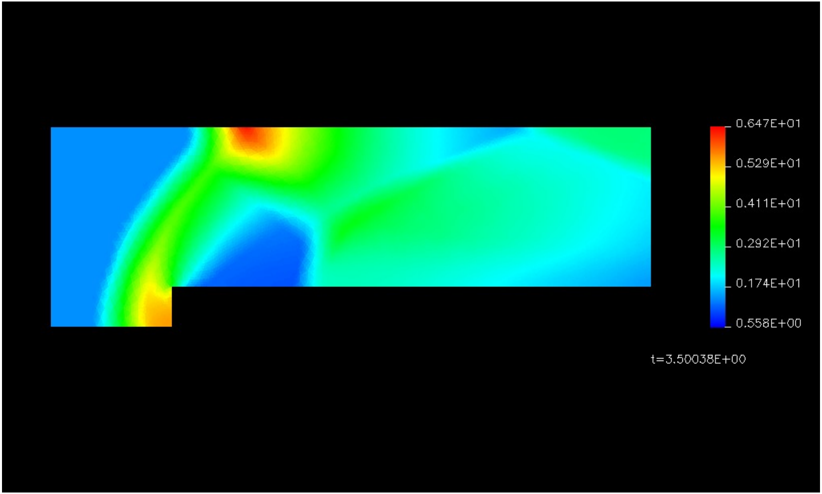
FFS - density distribution, t=2.5



FFS - density distribution, t=3.0



FFS - density distribution, t=3.5



FFS - density distribution, t=4.0