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High Resolution Schemes for Open Channel Flow

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Outline

- Mathematical models
- Finite volume methods and their properties
- Explicit x Implicit methods
- Semi-implicit upwind method
- Semi-implicit central-upwind method
- Numerical experiments

Fluid flow modelling

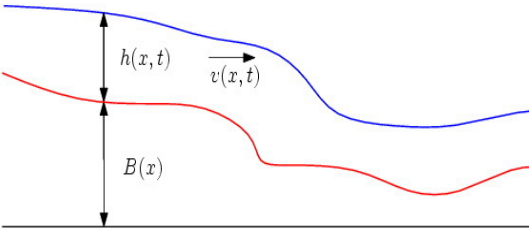
- Navier-Stokes equations
  One of the most general models for viscous compressible flow modelling
- Saint-Venant equations
  Modelling of inviscid incompressible flow, vertical component of acceleration is neglected
- Kinematic wave approximation
  Continuity relation and discharge rating curve

1D Shallow water equations

$$\begin{aligned} h_t + (hv)_x &= 0, \\ (hv)_t + \left(hv^2 + \tfrac{1}{2}gh^2\right)_x &= -ghB_x, \end{aligned} \tag{1}$$

where

- $h = h(x,t)$ ... unknown fluid depth
- $v = v(x,t)$ ... unknown horizontal velocity
- $B = B(x)$ ... elevation of the bottom
- $g = 9.81$ ... gravitational constant



Balance laws

Conservation law

$$\mathbf{u}_t + [\mathbf{f}(\mathbf{u}, x)]_x = \mathbf{0}, \tag{2}$$

- $\mathbf{u}$ ... unknown function
- $\mathbf{f}(\mathbf{u}, x)$ ... flux function

Balance law

$$\mathbf{u}_t + [\mathbf{f}(\mathbf{u}, x)]_x = \boldsymbol{\psi}(\mathbf{u}, x), \tag{3}$$

- $\boldsymbol{\psi}(\mathbf{u}, x)$ ... source term

Quasilinear form

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = \boldsymbol{\psi}(\mathbf{u}, x). \tag{4}$$

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Finite volume methods

Space and time discretization

$x_j = j\Delta x, t_n = n\Delta t, \qquad j, n \in \mathbb{Z}, n \geq 0,$
Finite volumes... $\langle x_{j-1/2}, x_{j+1/2} \rangle$.

Integral formulation of conservation law

$$\int_{x_1}^{x_2} \mathbf{u}(x, t_{n+1})\, dx - \int_{x_1}^{x_2} \mathbf{u}(x, t_n)\, dx + \int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{u}(x_2, t))\, dt - \int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{u}(x_1, t))\, dt = \mathbf{0}, \quad \forall (x_1, x_2) \times (t_n, t_{n+1}) \subset \mathbf{R} \times (0, T). \tag{5}$$

We use approximations of the integral averages of the unknown functions instead of the approximations of the unknown functions

$$\bar{\mathbf{U}}_j \approx \Delta x \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}(x, t_n) dx. \tag{6}$$

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- Semidiscrete method in the fluctuation form
  - $$\frac{d}{dt}\bar{\mathbf{U}}_j = -\frac{1}{\Delta x}[\mathbf{A}^-(\mathbf{U}^{\pm}_{j+1/2}) + \mathbf{A}(\mathbf{U}^{\pm}_j) + \mathbf{A}^+(\mathbf{U}^{\pm}_{j-1/2})] + \boldsymbol{\Psi}_j, \tag{8}$$
  - Source terms are subtracted from the flux difference

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It is recommended to construct a well balanced scheme

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Properties of the methods

Approximations should have the similar properties like exact solution. The properties result from decompositions of flux functions.

- Positive semidefiniteness
  $h \geq 0$
- Preserving steady states
  $\mathbf{u}_t = \mathbf{0}$ ($[\mathbf{f}(\mathbf{u}, x)]_x = \boldsymbol{\psi}(\mathbf{u}, x)$), special steady states
- Conservativity
  conservation law $\rightarrow$ conservative scheme
- TVD - total variation diminishing
  total variation of unknown functions $TV(U^n) = \sum_{j=-\infty}^{\infty} |U^n_{j+1} - U^n_j|$
- Numerical diffusion
  exact solution has no diffusion, higher in central schemes
- Formal order of accuracy
  can be influenced by the discontinuity of solution

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Explicit methods

- easy to implement
- low cost per time step
- time step is bounded by CFL stability condition
- inefficient for solution of stationary problems

Implicit methods

- uncoditionally stable (or stable over a wide range of time steps)
- difficult to implement
- high cost per time step
- insufficiently accurate for transient problems at large Δt
- problems with convergence of linear solvers as Δt increases

Semi-implicit upwind method - stability

Scheme in the conservative form

$$\frac{\mathbf{U}_j^{n+1}-\mathbf{U}_j^n}{\Delta t} = -\frac{1}{\Delta x}[(1-\theta)(\mathbf{F}_{j+1/2}^n-\mathbf{F}_{j-1/2}^n)+\theta(\mathbf{F}_{j+1/2}^{n+1}-\mathbf{F}_{j-1/2}^{n+1})]+(1-\theta)\mathbf{\Psi}_j^n+\theta\mathbf{\Psi}_j^{n+1}. \quad (10)$$

Stability

- $\theta = 0$ ... explicit scheme - CFL stability condition
- $\theta = 1$ ... implicit scheme - unconditionally stable

In the scalar case, the semi-implicit scheme ($0 < \theta < 1$) is TVD stable under the CFL condition

$$\mathrm{CFL} \leq \frac{1}{1-\theta}, \qquad \mathrm{CFL} = \frac{\Delta t}{\Delta x}\max_{p=1,2}|\lambda^p|. \quad (11)$$

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Semi-implicit upwind method - decomposition

Upwind scheme based on approximate Riemann solver
Decomposition of discontinuities of the unknown function

$$\Delta \mathbf{U}^n_{j+1/2} = \mathbf{U}^n_{j+1} - \mathbf{U}^n_j = \sum_{p=1}^{m} \alpha^{p,n}_{j+1/2} \mathbf{r}^{p,n}_{j+1/2}. \tag{12}$$

Define numerical fluxes

$$\mathbf{F}^n_{j+1/2} = \frac{1}{2}[\mathbf{f}(\mathbf{U}^n_j) + \mathbf{f}(\mathbf{U}^n_{j+1})] - \frac{1}{2}|\mathbf{A}^n_{j+1/2}|\Delta \mathbf{U}^n_{j+1/2}, \tag{13}$$

where $\mathbf{A}^n_{j+1/2}$ is the approximation of the Jacobian matrix and

$$|\mathbf{A}^n_{j+1/2}| = \mathbf{R}^n_{j+1/2}|\mathbf{\Lambda}^n_{j+1/2}|\mathbf{L}^n_{j+1/2}\mathbf{R}^n_{j+1/2}. \tag{14}$$

$\mathbf{R}^n_{j+1/2}$ ... matrix composed of the eigenvectors of $\mathbf{A}^n_{j+1/2}$
$\mathbf{\Lambda}^n_{j+1/2}$ ... diagonal matrix of the eigenvalues of $\mathbf{A}^n_{j+1/2}$
$\mathbf{L}^n_{j+1/2}$ ... flux limited matrix.

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Semi-implicit upwind method - linearization

$$\mathbf{L}^n_{j+1/2} = \mathbf{I} + \operatorname{diag}\left(\varphi(\mathbf{u})\left(1 - \min\left\{1, |\lambda^p_{j+1/2}|\frac{\Delta t}{\Delta x}\right\}\right)\right), \quad (15)$$

where $\varphi(\mathbf{u})$ is some limiter function.
If CFL > 1 then $\mathbf{L}^n_{j+1/2} = \mathbf{I}$ and the scheme is first order accurate.

Linearization

It used linearization for evaluating at the time layer t_{n+1}

$$\mathbf{f}(\mathbf{U}^{n+1}_j) \approx \mathbf{f}(\mathbf{U}^n_j) + \mathbf{A}^n_{j+1/2}(\mathbf{U}^{n+1}_j - \mathbf{U}^n_j) \quad (16)$$

$$\boldsymbol{\psi}(\mathbf{U}^{n+1}_j) \approx \boldsymbol{\psi}(\mathbf{U}^n_j) + \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{u}}(u^n_j)(\mathbf{U}^{n+1}_j - \mathbf{U}^n_j) \quad (17)$$

Semi-implicit upwind method - linearization

$$\mathbf{L}_{j+1/2}^{n} = \mathbf{I} + \operatorname{diag}\left(\varphi(\mathbf{u})\left(1 - \min\left\{1, |\lambda_{j+1/2}^{p}|\frac{\Delta t}{\Delta x}\right\}\right)\right), \tag{15}$$

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Semi-implicit upwind method - well-balancing

It is used upwind decomposition of the numerical flux. For preserving balancing property it is necessary decomposed source terms integral in a similar manner

$$\mathbf{\Psi}_j^n = \mathbf{\Psi}_{j+1/2}^{n,-} + \mathbf{\Psi}_{j-1/2}^{n,+}, \tag{18}$$

where

$$\mathbf{\Psi}_{j+1/2}^{n,\pm} = \frac{1}{2}(\mathbf{I} \pm \mathbf{A}_{j+1/2}^{-1}|\mathbf{A}_{j+1/2}|)\mathbf{\Psi}_{j+1/2}^{n} \tag{19}$$

Central-upwind scheme

Preserve only special steady states, where spatially derivatives of unknown functions (reconstructions) are equal to zero... define new unknown function for water level $c = h + B$

$$\begin{pmatrix} c \\ hv \end{pmatrix}_t + \begin{pmatrix} hv \\ \frac{(hv)^2}{c-B} + \frac{1}{2}g(c-B)^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ -g(c-B)B_x \end{pmatrix}. \tag{20}$$

Semidiscrete scheme

$$\frac{d}{dt}\mathbf{U}_j(t) = -\frac{\mathbf{F}^n_{j+1/2}(t) - \mathbf{F}^n_{j-1/2}(t)}{\Delta x} + \mathbf{\Psi}_j(t). \tag{21}$$

Numerical fluxes

$$\mathbf{F}^n_{j+1/2} = \frac{a^+_{j+1/2}\mathbf{f}(\mathbf{U}^-_{j+1/2}) - a^-_{j+1/2}\mathbf{f}(\mathbf{U}^+_{j+1/2})}{a^+_{j+1/2} - a^-_{j+1/2}} + \frac{a^+_{j+1/2}a^-_{j+1/2}}{a^+_{j+1/2} - a^-_{j+1/2}}\left[\mathbf{U}^+_{j+1/2} - \mathbf{U}^-_{j+1/2}\right], \tag{22}$$

where

$$\begin{aligned} a^+_{j+1/2} &= \max\left\{\lambda^N\left(\mathbf{f}'(\mathbf{U}^-_{j+1/2})\right), \lambda^N\left(\mathbf{f}'(\mathbf{U}^+_{j+1/2})\right), 0\right\}, \\ a^-_{j+1/2} &= \min\left\{\lambda^1\left(\mathbf{f}'(\mathbf{U}^-_{j+1/2})\right), \lambda^1\left(\mathbf{f}'(\mathbf{U}^+_{j+1/2})\right), 0\right\}. \end{aligned} \tag{23}$$

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Central-upwind scheme - "rest at lake"

Non-balanced scheme - "standard" discretization of the source term for example

$$\Psi_j^{(2)} = -gH_j^n \frac{B_{j+1} - B_{j-1}}{2\Delta x}. \tag{24}$$

Special steady state ($hv = 0, h + B = \text{ konst.}$). Flux difference

$$\begin{aligned} -\frac{F^{(2)}_{j+1/2} - F^{(2)}_{j-1/2}}{\Delta x} &= -\frac{1}{2\Delta x} g \left(\left(C_{j+1/2} - B(x_{j+1/2})\right)^2 - \left(C_{j-1/2} - B(x_{j-1/2})\right)^2 \right) = \\ &= g \frac{B(x_{j+1/2}) - B(x_{j-1/2})}{\Delta x} \cdot \frac{C_{j+1/2} - B(x_{j+1/2}) + C_{j-1/2} - B(x_{j-1/2})}{2}. \end{aligned} \tag{25}$$

Steady state means $\mathbf{u}_t = 0$... well balancing... discretization of the source terms

$$\Psi_j^{(2)} = -g \frac{B(x_{j+1/2}) - B(x_{j-1/2})}{\Delta x} \cdot \frac{\left(C^-_{j+1/2} - B(x_{j+1/2})\right) + \left(C^+_{j-1/2} - B(x_{j-1/2})\right)}{2}. \tag{26}$$

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$$\begin{aligned} -\frac{F^{(2)}_{j+1/2} - F^{(2)}_{j-1/2}}{\Delta x} &= -\frac{1}{2\Delta x} g \left(\left(C_{j+1/2} - B(x_{j+1/2})\right)^2 - \left(C_{j-1/2} - B(x_{j-1/2})\right)^2 \right) = \\ &= g \frac{B(x_{j+1/2}) - B(x_{j-1/2})}{\Delta x} \cdot \frac{C_{j+1/2} - B(x_{j+1/2}) + C_{j-1/2} - B(x_{j-1/2})}{2}. \end{aligned} \tag{25}$$

Steady state means $\mathbf{u}_t = 0$... well balancing... discretization of the source terms

$$\Psi_j^{(2)} = -g \frac{B(x_{j+1/2}) - B(x_{j-1/2})}{\Delta x} \cdot \frac{\left(C^-_{j+1/2} - B(x_{j+1/2})\right) + \left(C^+_{j-1/2} - B(x_{j-1/2})\right)}{2}. \tag{26}$$

Central-upwind scheme - "rest at lake"

Non-balanced scheme - "standard" discretization of the source term for example

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Central-upwind scheme - reconstruction

Polynomial TVD reconstruction

$$\begin{aligned} \mathbf{U}^{+,n}_{j+1/2} &= \mathbf{U}^n_{j+1} - \left(1 - \min\left\{1, |\lambda^{\max}_{j+1/2}| \frac{\Delta t}{\Delta x}\right\}\right) \frac{\Delta x}{2} (\mathbf{U}_x)^n_{j+1} \\ \mathbf{U}^{-,n}_{j+1/2} &= \mathbf{U}^n_{j} + \left(1 - \min\left\{1, |\lambda^{\max}_{j+1/2}| \frac{\Delta t}{\Delta x}\right\}\right) \frac{\Delta x}{2} (\mathbf{U}_x)^n_{j}, \end{aligned} \quad (27)$$

where $(\mathbf{U}_x)^n_j$ is defined

$$(\mathbf{U}_x)^n_j = \operatorname{minmod}\left(\frac{\mathbf{U}^n_j - \mathbf{U}^n_{j-1}}{\Delta x}, \frac{\mathbf{U}^n_{j+1} - \mathbf{U}^n_j}{\Delta x}\right), \quad (28)$$

The minmod function is minmod (a, b) is defined as follows

$$\operatorname{minmod}(a, b) = \frac{1}{2}\left[\operatorname{sgn}(a) + \operatorname{sgn}(b)\right] \cdot \min\left(|a|, |b|\right). \quad (29)$$

Reconstruction at the time layer t_{n+1} use the same derivative as at the time layer t_n

Semi-implicit central-upwind scheme

Semi-implicit scheme

$$\frac{\mathbf{U}_j^{n+1} - \mathbf{U}_j^n}{\Delta t} = -\frac{1}{\Delta x}[(1-\theta)(\mathbf{F}_{j+1/2}^n - \mathbf{F}_{j-1/2}^n) + \theta(\mathbf{F}_{j+1/2}^{n+1} - \mathbf{F}_{j-1/2}^{n+1})] + (1-\theta)\mathbf{\Psi}_j^n + \theta\mathbf{\Psi}_j^{n+1}. \tag{30}$$

where

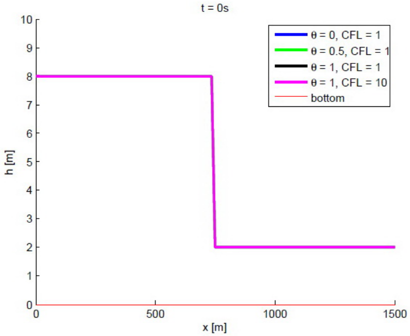
$$\mathbf{F}_{j+1/2}^{n+1} = \frac{a_{j+1/2}^{+,n}\mathbf{f}(\mathbf{U}_{j+1/2}^{-,n+1}) - a_{j+1/2}^{-,n}\mathbf{f}(\mathbf{U}_{j+1/2}^{+,n+1})}{a_{j+1/2}^{+,n} - a_{j+1/2}^{-,n}} + \frac{a_{j+1/2}^{+,n}a_{j+1/2}^{-,n}}{a_{j+1/2}^{+,n} - a_{j+1/2}^{-,n}}\left[\mathbf{U}_{j+1/2}^{+,n+1} - \mathbf{U}_{j+1/2}^{-,n+1}\right], \tag{31}$$

Linearization of the flux function

$$\mathbf{f}(\mathbf{U}_j^{n+1}) \approx \mathbf{f}(\mathbf{U}_j^n) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(u_j^n)(\mathbf{U}_j^{n+1} - \mathbf{U}_j^n) \tag{32}$$

Numerical experiment 1 - Central-upwind scheme

Numerical viscosity

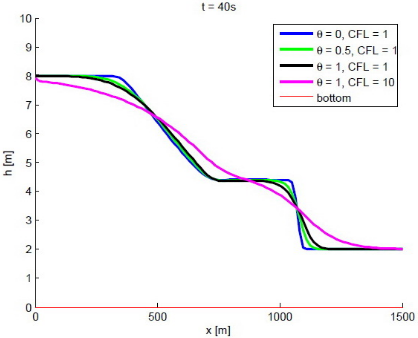


$$h(x,0) + B(x) = \begin{cases} 8, & x \in \langle 0, 750\rangle, \\ 2, & x \in (750, 1500\rangle. \end{cases} \quad , v(x,0) = 0 \tag{33}$$

$$q(0,t) = \text{const.} \tag{34}$$

Numerical experiment 1 - Central-upwind scheme

Numerical viscosity



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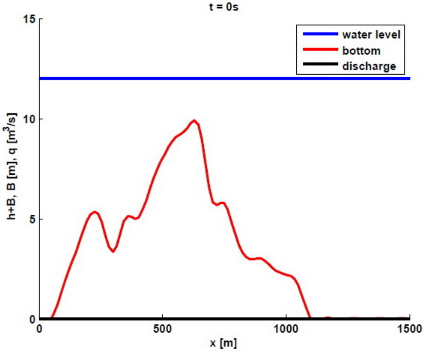
Numerical experiment 2 - Central-upwind scheme

Preserving steady state - balanced implicit scheme

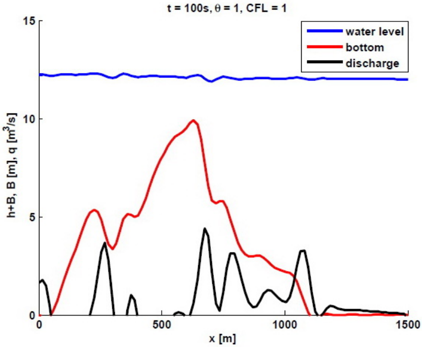


$$h(x,0) + B(x) = 12, v(x,0) = 0 \quad (35)$$

$$q(0,t) = \text{const.} \quad (36)$$

Numerical experiment 2 - Central-upwind scheme

Preserving steady state - balanced implicit scheme



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Numerical experiment 2 - Central-upwind scheme

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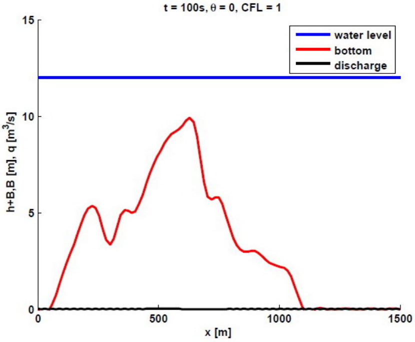

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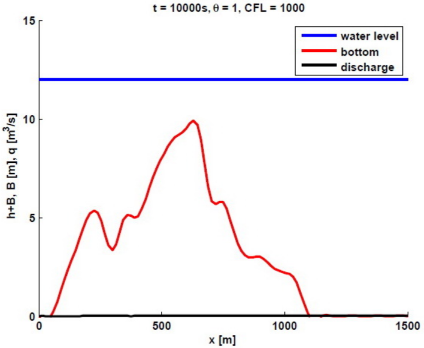

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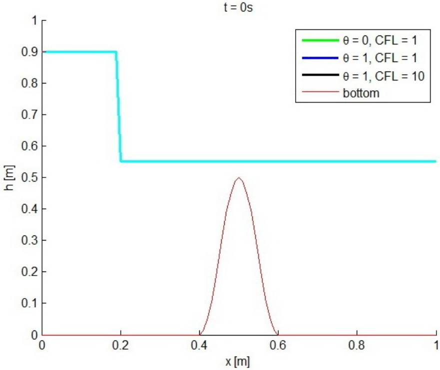
Numerical experiment 3 - Central-upwind scheme

General steady state

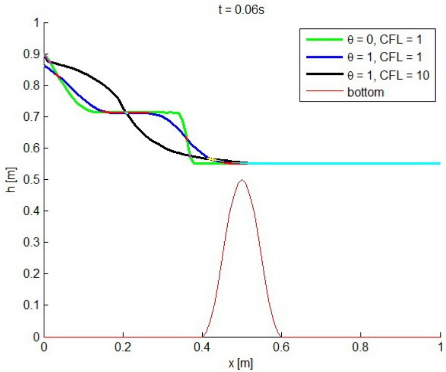


$$h(x,0) + B(x) = \begin{cases} 0.9, & x \in \langle 0, 0.2\rangle, \\ 0.55, & x \in (0.2, 1\rangle. \end{cases}, v(x,0) = \begin{cases} 0.1 & x \in \langle 0, 0.2\rangle, \\ 0 & x \in (0.2, 1\rangle. \end{cases} \tag{37}$$

$$q(0,t) = \text{const.} \tag{38}$$

Numerical experiment 3 - Central-upwind scheme

General steady state



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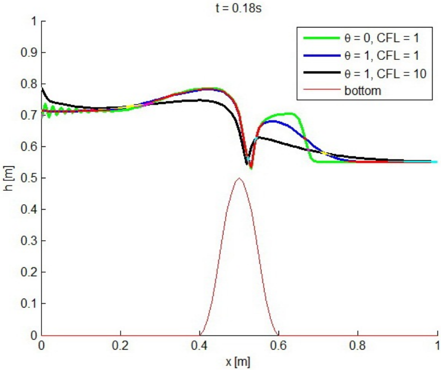
General steady state



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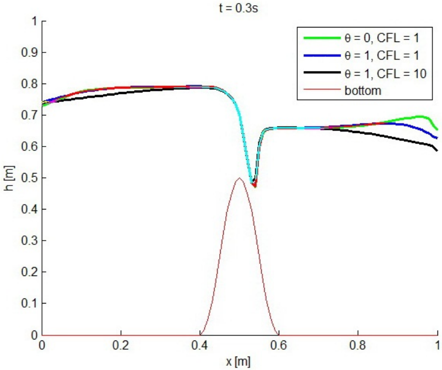

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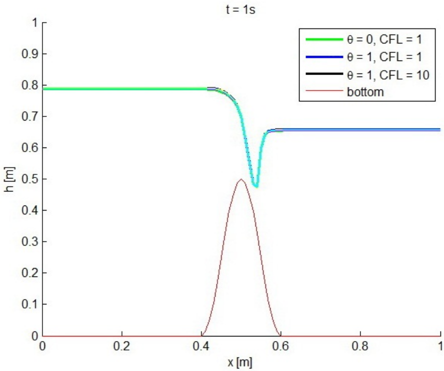

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Conclusion

- Numerical diffusion
- Efficient for steady state problems
- Discontinuities - small CFL
- Extension to two-dimensional case
- Problems with dry states

References

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