Some remarks on averaging in the BDDC method

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Outline

BDDC (Balanced Domain Decomposition Method)

Choice of the averaging operator

Numerical results

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Brief overview of BDDC method

- **B**alancing **D**omain **D**ecomposition by **C**onstraints
- 2003 C. Dohrmann (Sandia), theory with J. Mandel (UCD)
- nonoverlapping primary domain decomposition method
- equivalent to FETI-DP [Mandel, Dohrmann, Tezaur 2005]

Abstract problem

$$\mathbf{Au} = \mathbf{f}$$

- $A : U \to U', \quad u \in U, \quad f \in U'$
- dim $U < \infty$
- A symmetric positive definite

PCG with a preconditioner M

- the operator MA has good spectral properties
  – M can be regarded as an approximation of A^{-1}
- a preconditioned residual $p = Mr$
  is "cheap" to obtain for any given r

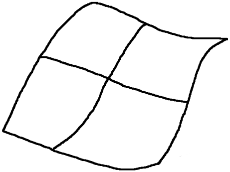
Abstract BDDC preconditioner

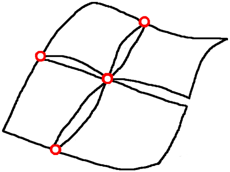
$$
\begin{array}{ccccccc}
 & & U & \overset{A}{\longrightarrow} & U' & & \\
E\uparrow & R\downarrow & & & \uparrow R^T & \downarrow E^T \\
 & & \widetilde{W} & \overset{\widetilde{A}^{-1}}{\longleftarrow} & \widetilde{W}' & &
\end{array}
$$

- $R \ldots$ operator of injection
- $A = R^T \widetilde{A} R$, $\quad A, \widetilde{A} \ldots$ symmetric positive definite
- $E \ldots$ projection (averaging), $\quad E\,R \;=\; I$
- $M := E\widetilde{A}^{-1}E^T \approx A^{-1}$
- $cond\,(MA) \leq ||RE||^2_{\widetilde{A}}$ [Mandel, Sousedík 2007]

BDDC in FEM context

$$Au = f \qquad \overset{E}{\longleftarrow} \qquad \widetilde{A}\widetilde{u} = \widetilde{f}$$





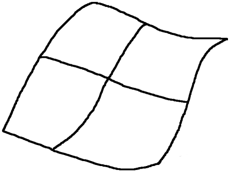
$$\mathbf{U} \subset \widetilde{\mathbf{W}}$$

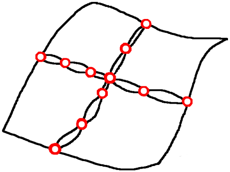
U
continuous
at all dofs
across interface

$\widetilde{\mathbf{W}}$
continuous
only at selected
"coarse" dofs

BDDC in FEM context

$$Au = f \quad \overset{E}{\longleftarrow} \quad \widetilde{A}\widetilde{u} = \widetilde{f}$$

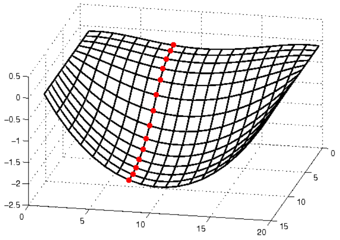




$$\mathbf{U} \quad \subset \quad \widetilde{\mathbf{W}}$$

$\mathbf{U}$		$\widetilde{\mathbf{W}}$
continuous at all dofs across interface	$\subset$	continuous only at selected "coarse" dofs

Model: 2D Poisson problem $Ku = f$, 2 subdomains



Schur complement: $Ku = f \;\Rightarrow\; \widehat{S}\widehat{u} = \widehat{g}$

$$Ku = \begin{bmatrix} K_{oo}^1 & & \widehat{K}_{or}^1 \\ & K_{oo}^2 & \widehat{K}_{or}^2 \\ \widehat{K}_{ro}^1 & \widehat{K}_{ro}^2 & \widehat{K}_{rr} \end{bmatrix} \begin{bmatrix} u_o^1 \\ u_o^2 \\ \widehat{u} \end{bmatrix} = \begin{bmatrix} f_o^1 \\ f_o^2 \\ \widehat{f} \end{bmatrix} = f$$

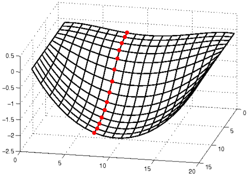
elimination of interior nodes:

$$\begin{bmatrix} K_{oo}^1 & & \widehat{K}_{or}^1 \\ & K_{oo}^2 & \widehat{K}_{or}^2 \\ & & \widehat{S} \end{bmatrix} \begin{bmatrix} u_o^1 \\ u_o^2 \\ \widehat{u} \end{bmatrix} = \begin{bmatrix} f_o^1 \\ f_o^2 \\ \widehat{g} \end{bmatrix}$$

$$\widehat{S} = \widehat{K}_{rr} - \sum \widehat{K}_{ro}^i (K_{oo}^i)^{-1} \widehat{K}_{or}^i$$

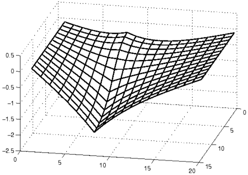
$$\widehat{g} = \widehat{f} - \sum \widehat{K}_{ro}^i (K_{oo}^i)^{-1} f_o^i$$

Schur complement: $\quad Ku = f \;\Rightarrow\; \widehat{S}\,\widehat{u}_r = \widehat{g}$

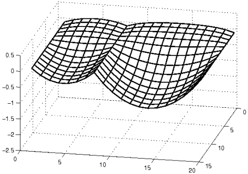


$$
\begin{bmatrix} K_{oo}^1 & & \widehat{K}_{or}^1 \\ & K_{oo}^2 & \widehat{K}_{or}^2 \\ & & \widehat{S} \end{bmatrix}
\begin{bmatrix} u_o^1 \\ u_o^2 \\ \widehat{u} \end{bmatrix}
=
\begin{bmatrix} f_o^1 \\ f_o^2 \\ \widehat{g} \end{bmatrix}
$$

$$
\sim \quad \widehat{S}\,\widehat{u} = \widehat{g}\,, \qquad K_{oo}^i u_o^i = f_o^i - \widehat{K}_{or}^i\,\widehat{u}
$$



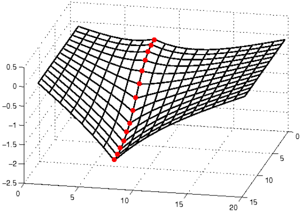
$$
\begin{bmatrix} K_{oo}^1 & & \widehat{K}_{or}^1 \\ & K_{oo}^2 & \widehat{K}_{or}^2 \\ & & \widehat{S} \end{bmatrix}
\begin{bmatrix} u_{or}^1 \\ u_{or}^2 \\ \widehat{u} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \widehat{g} \end{bmatrix}
$$



$$
\begin{bmatrix} K_{oo}^1 & & \widehat{K}_{or}^1 \\ & K_{oo}^2 & \widehat{K}_{or}^2 \\ & & \widehat{S} \end{bmatrix}
\begin{bmatrix} u_{oo}^1 \\ u_{oo}^2 \\ 0 \end{bmatrix}
=
\begin{bmatrix} f_o^1 \\ f_o^2 \\ 0 \end{bmatrix}
$$

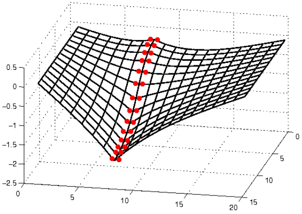
Dividing the Schur complement into subdomains

U



$R \downarrow \quad \uparrow E$

$\widehat{W}$



$$\widehat{S}\,\widehat{u} = \widehat{g}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

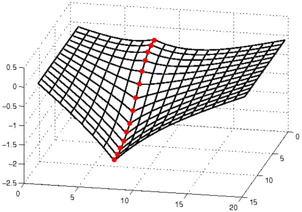
$$R\,\widehat{u} = u_r$$
$$\widehat{S} = R^T S\, R$$
$$\widehat{g} = R^T g$$

$$?\; S\, u_r = g$$

$$S = \begin{bmatrix} S^1 & 0 \\ 0 & S^2 \end{bmatrix}$$

Dividing the Schur complement into subdomains

U



$$\widehat{S}\,\widehat{u} = \widehat{g}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

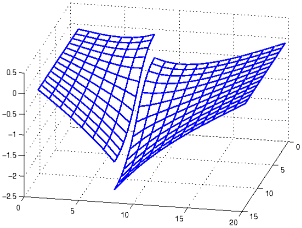
$R \downarrow \quad \uparrow E$

$$R\,\widehat{u} = u_r$$

$$\widehat{S} = R^T S\, R$$

$$\widehat{g} = R^T g$$

W

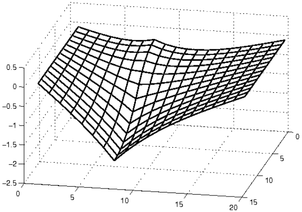


$$S\,u = g$$

$$S = \begin{bmatrix} S^1 & 0 \\ 0 & S^2 \end{bmatrix}$$

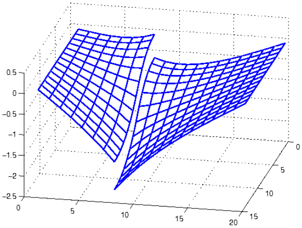
Primal methods (one level)

U



$R \downarrow \quad \uparrow E$

W



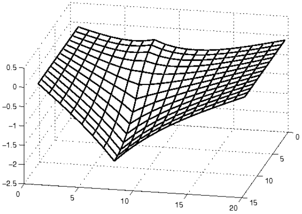
$\widehat{u}^{(0)}$; for $k := 0, 1, 2, \ldots$:

1. $\widehat{r}^{(k)} := \widehat{g} - R^T S \, R \, \widehat{u}^{(k)}$
   $|\,\widehat{r}^{(k)}\,| < \epsilon \Rightarrow$ end
2. $\Delta r^{(k)} := E^T \widehat{r}^{(k)}$
3. $\Delta {u_r}^{(k)} := S^{-1} \Delta r^{(k)}$
4. $\Delta \widehat{u}^{(k)} := E \, \Delta u_r^{(k)}$
5. $\widehat{u}^{(k+1)} := \widehat{u}^{(k)} + \Delta \widehat{u}^{(k)}$

$$\widehat{u}^{(k+1)} = \widehat{u}^{(k)} + E \, S^{-1} E^T \widehat{r}^{(k)}$$

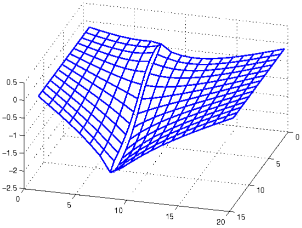
Primal methods – BDDC

U



$R \downarrow \quad \uparrow E$

$\widetilde{W}$



$\widehat{u}^{(0)}$; for $k := 0, 1, 2, \ldots$:

1. $\widehat{r}^{(k)} := \widehat{g} - R^T S R \widehat{u}^{(k)}$
   $|\widehat{r}^{(k)}| < \epsilon \Rightarrow$ end
2. $\Delta r^{(k)} := E^T \widehat{r}^{(k)}$
3. $\Delta {u_r}^{(k)} := \widetilde{S}^{-1} \Delta r^{(k)}$
4. $\Delta \widehat{u}^{(k)} := E \Delta u_r^{(k)}$
5. $\widehat{u}^{(k+1)} := \widehat{u}^{(k)} + \Delta \widehat{u}^{(k)}$

$$\widehat{u}^{(k+1)} = \widehat{u}^{(k)} + E \widetilde{S}^{-1} E^T \widehat{r}^{(k)}$$

Relationship between primal and dual methods

Let $$B_D^T B + RE = I\,.$$

Then the primal and the corresponding dual preconditioned systems are spectrally equivalent.

[Mandel, Dohrmann, Tezaur 2005]

B . . . the jump operator
B_D . . . the weighted jump operator

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BDDC - operators R, E : an example

4 interface nodes, 2 coarse, E – arithmetic average

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\widehat{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad \mathbf{R}\,\widehat{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ u_3 \\ u_4 \end{bmatrix} \quad \mathbf{E}\,\mathbf{u} = \begin{bmatrix} \left(u_1^1 + u_1^2\right)/2 \\ \left(u_2^1 + u_2^2\right)/2 \\ u_3 \\ u_4 \end{bmatrix}$$

BDDC - operators R, E (2 subdomains)

$$\mathbf{R} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} \mathbf{A} & \mathbf{I}-\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \widetilde{\mathbf{S}} = \begin{bmatrix} \mathbf{S}^1 & \mathbf{0} & \mathbf{S}^1_{\mathrm{rc}} \\ \mathbf{0} & \mathbf{S}^2 & \mathbf{S}^2_{\mathrm{rc}} \\ \mathbf{S}^1_{\mathrm{cr}} & \mathbf{S}^2_{\mathrm{cr}} & \mathbf{S}_{\mathrm{cc}} \end{bmatrix}$$

$$\widehat{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_{\mathrm{r}} \\ \mathbf{u}_{\mathrm{c}} \end{bmatrix} \quad \mathbf{R}\widehat{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_{\mathrm{r}} \\ \mathbf{u}_{\mathrm{r}} \\ \mathbf{u}_{\mathrm{c}} \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}^1_{\mathrm{r}} \\ \mathbf{u}^2_{\mathrm{r}} \\ \mathbf{u}_{\mathrm{c}} \end{bmatrix} \quad \mathbf{E}\mathbf{u} = \begin{bmatrix} \mathbf{A}\mathbf{u}^1_{\mathrm{r}} \\ (\mathbf{I}-\mathbf{A})\mathbf{u}^2_{\mathrm{r}} \\ \mathbf{u}_{\mathrm{c}} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & \dots & \alpha_n \end{bmatrix} \qquad \alpha_i = s^1_{\mathrm{ii}} / (s^1_{\mathrm{ii}} + s^2_{\mathrm{ii}}) = s^1_{\mathrm{ii}} / \widehat{s}_{\mathrm{ii}}$$

Minimization of *RE* in energetic norm

$$||\mathbf{RE}\,\mathbf{u}||_S^2 \longrightarrow \min \quad \text{with respect to } \alpha_i \quad \text{results in}$$

$$\begin{bmatrix} \hat{s}_{11} d_1 d_1 & \hat{s}_{12} d_1 d_2 & \dots & \hat{s}_{1n} d_1 d_n \\ \dots & \dots & \dots & \dots \\ \hat{s}_{i1} d_i d_1 & \hat{s}_{i2} d_i d_2 & \dots & \hat{s}_{in} d_i d_n \\ \dots & \dots & \dots & \dots \\ \hat{s}_{n1} d_n d_1 & \hat{s}_{n2} d_n d_2 & \dots & \hat{s}_{nn} d_n d_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \dots \\ \dots \\ \dots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \sum_j s^1_{1j} d_j d_1 \\ \dots \\ \sum_j s^1_{ij} d_j d_i \\ \dots \\ \sum_j s^1_{nj} d_j d_n \end{bmatrix},$$

where $\mathbf{d} = \mathbf{u}_{\mathrm{r}}^2 - \mathbf{u}_{\mathrm{r}}^1 = (d_1, d_2, \dots d_n)^T$ is the jump.

Simplifying choices:

$$\mathbf{d} = (0, \dots 0, d_i, 0 \dots 0) \quad \Longrightarrow \quad \alpha_i = s^1_{\mathrm{ii}} / \left(s^1_{\mathrm{ii}} + s^2_{\mathrm{ii}}\right) = s^1_{\mathrm{ii}} / \hat{s}_{\mathrm{ii}}$$

$$\alpha_1 = \dots = \alpha_n = \alpha \quad \Longrightarrow \quad \alpha = \mathbf{d}^T \mathbf{S}^1 \mathbf{d} / \mathbf{d}^T (\mathbf{S}^1 + \mathbf{S}^2)\, \mathbf{d}$$

Choice of E

In general, let us try the choice

$$\alpha = \mathbf{v}^T \mathbf{S}^1 \mathbf{v} / \mathbf{v}^T (\mathbf{S}^1 + \mathbf{S}^2) \mathbf{v}$$

for different test vectors $\mathbf{v}$:

$$\mathbf{v} = \mathbf{e}_i = (0, \ldots 0, 1, 0 \ldots 0) \quad \Longrightarrow \quad \alpha_i = s^1_{\mathrm{ii}} / (s^1_{\mathrm{ii}} + s^2_{\mathrm{ii}})$$

$$\mathbf{v} = (1, \ldots 1) \quad \Longrightarrow \quad \alpha = \sum_{i,j} s^1_{\mathrm{ij}} / \sum_{i,j} (s^1_{\mathrm{ij}} + s^2_{\mathrm{ij}})$$

$$\mathbf{v} = (1, \ldots 1, 0, \ldots, 0)$$

$\ldots$

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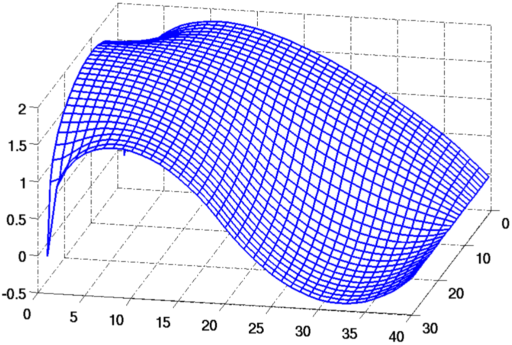
Test problem: 2D Poisson equation

bilinear elements 40 (left-right) x 30 (back-front)

2 subdomains in left-right direction: either 1:1, or 2:1

without PCG



0c . . . no coarse degrees of freedom
2c . . . 2 coarse nodes at interface

Subdomains 1:1, errors in 5 iterations

	$\alpha = 1/2$	α_i diag. Sch.	$v = (1, ..1)$	$v = d$	α
0 c		0.4997-9	0.276		
	1.5001	1.4966	0.4235	1.5001	0.5
	0.3872	0.3854	0.0806	0.0001	0.276
	0.0999	0.0992	0.0153	2e-06	0.424
	0.0258	0.0255	0.0029	1e-09	0.492
	0.0066	0.0066	0.0006	4e-15	0.276
2 c		0.49969-74	0.397		
	0.7349	0.7332	0.2402	0.7349	0.5
	0.0929	0.0925	0.0140	0.0211	0.376
	0.0117	0.0117	0.0008	0.0012	0.376
	0.0015	0.0015	5e-05	7e-05	0.376
	0.0002	0.0002	3e-06	4e-06	0.376

Subdomains 2:1, errors in 5 iterations

	$\alpha = 1/2$	α_i diag. Sch.	$v = (1, ..1)$	$v = d$	α
0 c		0.4993-7	0.191		
	1.7909	1.7851	0.9373	1.7909	0.5
	1.1010	1.0938	0.3034	0.0022	0.193
	0.6769	0.6702	0.0982	0.0004	0.273
	0.4161	0.4107	0.0318	7e-07	0.475
	0.2558	0.2517	0.0103	4e-11	0.191
2 c		0.4993-6	0.341		
	0.8663	0.8635	0.2690	0.8663	0.5
	0.2576	0.2560	0.0302	0.0476	0.316
	0.0766	0.0759	0.0035	0.0056	0.314
	0.0227	0.0225	0.0004	0.0007	0.314
	0.0068	0.0067	5e-05	8e-05	0.314

Conclusions

- Energetic norm of projection *RE* was analyzed,
- more general form of operator *E* was proposed,
- 4 choices of operator *E* were compared:

◦ $\alpha = 1/2$

◦ $\alpha = \mathbf{v}^T \mathbf{S}^1 \mathbf{v} / \mathbf{v}^T (\mathbf{S}^1 + \mathbf{S}^2) \mathbf{v}$

  - $\mathbf{v} = \mathbf{e}_i = (0, \ldots 0, 1, 0 \ldots 0)$ (fract. of Schur diag. entries)
  - $\mathbf{v} = (1, \ldots 1)$
  - $\mathbf{v} = \mathbf{d} = \mathbf{u}_\mathrm{r}^2 - \mathbf{u}_\mathrm{r}^1$

- more research and tests are needed.