Analysis of a higher order semiimplicit BDF for semilinear problems

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Introduction

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- model problem:
  scalar nonstationary nonlinear convection-diffusion equation,
- space semi-discretization (e.g., DGFEM),
- suitable time discretization (e.g. BDF)

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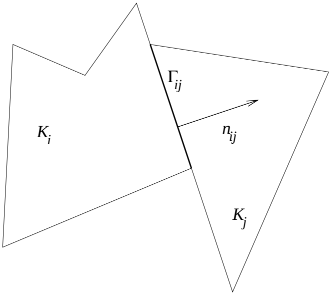
$$u\Big|_{\partial\Omega \times (0,T)} = u_D,$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

Triangulations



Space settings

- Let $V \subset L^2(\Omega)$ be a sufficiently regular space for exact solution

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- $V_{h,p}$ be the space of piecewise polynomials up to degree p

Notation

- elements K
- edges $\Gamma_h = \bigcup_K \partial K$
- arbitrary but fixed normals $\mathbf{n}$ to edges Γ_h
- for $v \in V_{h,p}$, $x \in \Gamma_h$ we set $v_L(x) = \lim_{\delta \to 0+} v(x - \delta \mathbf{n})$ and $v_R = \lim_{\delta \to 0+} v(x + \delta \mathbf{n})$
- for $v \in V_{h,p}$, $x \in \Gamma_h$ we set $[v] = v_L - v_R$ and $\langle v \rangle = \frac{v_L + v_R}{2}$

Diffusive form A_h

$$A_h(u,w) = \varepsilon \sum_K \int_K \nabla u \cdot \nabla w \,\mathrm{d}x$$
$$-\varepsilon \int_{\Gamma_h} \Big(\langle \nabla u \rangle \cdot \mathbf{n}[w] + \langle \nabla w \rangle \cdot \mathbf{n}[u] \Big) \,\mathrm{d}S + \varepsilon \int_{\Gamma_h} \sigma [u]\,[w] \,\mathrm{d}S,$$

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- $\|R_h v - v\| \leq Ch^{p+1}$

Convective form b_h

$$b_h(u,w) = \int_{\Gamma_h} H(u_L, u_R, \mathbf{n})\,[w]\,\mathrm{d}S - \sum_K \int_K \sum_{s=1}^{d} f_s(u)\,\frac{\partial w}{\partial x_s}\,\mathrm{d}x,$$

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- $b_h(v, w) - b_h(R_h v, w) \le Ch^{p+1}|||w|||$

Source form ℓ_h

$$\ell_h(w)\,(t) = (g(t), w) + \varepsilon \int_{\partial\Omega} (-\nabla w \cdot \mathbf{n}\, u_D + \sigma u_D\, w)\, \mathrm{d}S.$$

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- let $u_h(t_s) = u_h^s \approx U^s \in V_{h,p}$ for $s = 0, \ldots, r$

Euler method

- Backward Euler method linearized by Forward Euler method

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$$(U^{s+1} - U^s, v) + \tau\varepsilon A_h(U^{s+1}, v) + \tau b_h(U^s, v) = \tau\ell(v)$$

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$$\begin{aligned} \|e\|^2_{h,\tau,L^\infty(L^2)} &= O(h^{2p} + \tau^2), \\ \|e\|^2_{h,\tau,L^2(H^1)} &= O(h^{2p} + \tau^2), \end{aligned}$$

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$$\begin{aligned} \|e\|^2_{h,\tau,L^\infty(L^2)} &= O(h^{2p+2} + \tau^{2m}), \\ \|e\|^2_{h,\tau,L^2(H^1)} &= O(h^{2p+2} + \tau^{2m}), \end{aligned}$$

BDF coefficients-order conditions

$$\begin{aligned}
\sum_{j=0}^{m} \alpha_j &= 0 \\
\sum_{j=0}^{m} \alpha_j (m-j) &= -1 \\
\sum_{j=0}^{m} \alpha_j (m-j)^s &= 0 \quad s = 2, \ldots, m
\end{aligned}$$

BDF coefficients

$$
\begin{aligned}
\alpha_j &= (-1)^{m-j} \frac{\binom{m}{j}}{m-j} \quad j = 0, \ldots, m-1 \\
\alpha_m &= \sum_{j=1}^{m} \frac{1}{j} \\
\beta_j &= -\alpha_j (m-j) = (-1)^{m-j-1} \binom{m}{j}
\end{aligned}
$$

Technique of the proof

- we divide the error

$$e^s = U^s - u^s = \underbrace{U^s - R_h u^s}_{:=\xi^s} + \underbrace{R_h u^s - u^s}_{:=\eta^s}$$

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- it is sufficient to estimate $\|\xi^s\|$

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- substracting equation for weak solution from equation for discrete solution we obtain

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$$
\begin{aligned}
&\left(\sum_{j=0}^{m} \alpha_j \xi^{s+j}, v\right) + \tau\varepsilon A_h(\xi^{s+m}, v) \\
&= \left(\tau \frac{\partial u}{\partial t}(t_{s+m}) - \sum_{j=0}^{m} \alpha_j u^{s+j} - \sum_{j=0}^{m} \alpha_j \eta^{s+j}, v\right) \\
&\quad + \tau \left(b_h(u^{s+m}, v) - b_h(\sum_{j=0}^{m-1} \beta_j U^{s+j}, v) \right)
\end{aligned}
$$

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$$\sum_{s=0}^{n-m}\left(\sum_{j=0}^{m}\alpha_j\xi^{s+j},\gamma_{n-m-s}\xi^n\right)+\tau\varepsilon A_h(\xi^{s+m},\gamma_{n-m-s}\xi^n)$$
$$=\|\xi^n\|^2+\left(\sum_{s=0}^{m-1}\sum_{j=0}^{s}\alpha_j\gamma_{n-m-s+j}\xi^s,\xi^n\right)$$

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$$=\|\xi^n\|^2+\left(\sum_{s=0}^{m-1}\sum_{j=0}^{s}\alpha_j\gamma_{n-m-s+j}\xi^s,\xi^n\right)$$

- $\|\gamma_j\| \leq C$

Technique of the proof

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$$= \|\xi^n\|^2 + \left(\sum_{s=0}^{m-1}\sum_{j=0}^{s}\alpha_j\gamma_{n-m-s+j}\xi^s, \xi^n\right)$$

- $\|\gamma_j\| \leq C$
- $\tau\varepsilon\sum_{j=0}^{\infty}|||\gamma_j v|||^2 \leq C\|v\|^2$

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- $$\|\xi^n\|^2 \le C(\tau^m + h^{p+1} + \sum_{j=0}^{m-1} \|\xi^j\|^2) + \frac{C}{\varepsilon} \sum_{j=0}^{n-1} \|\xi^j\|^2$$

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- Gronwall lemma

Conclusion

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- IMEX BDF-DGFE discretization discussed

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- scalar nonstationary nonlinear convection-diffusion equation
- IMEX BDF-DGFE discretization discussed
- error estimates derived

Thank you for your attention.

Numerical example1

$$
\begin{aligned}
t &\in [0,1] \\
y'(t) &= -\varepsilon y(t) + y^2(t) \\
y(0) &= 1
\end{aligned}
$$

Numerical example1

$\tau = \frac{1}{20*2^m}$ m	BDF (one-leg) $\|\|e\|\|_{L^\infty}$	2^{nd} order rate	BDF (IMEX) $\|\|e\|\|_{L^\infty}$	2^{nd} order rate
1	3.2924	-	4.6737	-
2	1.2668	1.3780	2.0487	1.1898
3	0.3975	1.6723	0.7080	1.5329
4	0.1108	1.8425	0.2086	1.7629
5	0.0292	1.9259	0.0564	1.8866
6	0.0075	1.9647	0.0145	1.9461
7	0.0019	1.9829	0.0037	1.9740
8	0.0005	1.9916	0.0009	1.9873

Numerical example1

$\tau = \frac{1}{20*2^m}$ m	Runge–Kutta $\|e\|_{L^\infty}$	2^{nd} order rate	Time DG $\|e\|_{L^\infty}$	2^{nd} order rate
1	1.3770	-	2.7966	-
2	0.4270	1.6893	0.9946	1.4915
3	0.1173	1.8634	0.2912	1.7722
4	0.0306	1.9413	0.0774	1.9116
5	0.0078	1.9739	0.0198	1.9665
6	0.0020	1.9879	0.0050	1.9866
7	0.0005	1.9942	0.0013	1.9942
8	0.0001	1.9972	0.0003	1.9973

Numerical example2

$$\Omega = (0,1) \times (0,1), \quad T = 1, \quad \varepsilon = 0.02$$

$$f_s(u) = \frac{u^2}{2}, \quad s = 1,2$$

$$u = 16\frac{\exp(10t) - 1}{\exp(10) - 1}xy(1-x)(1-y)$$

Numerical example2

$\tau = \frac{0.4}{2^m}$ m	BDF $\lVert e \rVert_{L^\infty(L^2)}$	2^{nd} order rate	Time DG $\lVert e \rVert_{L^\infty(L^2)}$	2^{nd} order rate
1	$3.251E-01$	-	$3.650E-02$	-
2	$1.098E-01$	1.566	$1.899E-02$	0.943
3	$3.398E-02$	1.693	$6.421E-03$	1.564
4	$9.810E-03$	1.792	$1.802E-03$	1.833
5	$2.658E-03$	1.884	$4.717E-04$	1.934
6	$6.943E-04$	1.937	$1.203E-04$	1.971

Time discontinuous Galerkin

- $V_{h,\tau}^{p,q} \equiv \{v : v/_{(t_{s-1},t_s)}(t,x) = \sum_{j=0}^{q} y_j(x)t^j,\ \ y_j \in V_{h,p}\}$

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- $v_{\pm}^{s} \equiv \lim_{t\to t_s\pm} v(t), \quad \{v\}_s \equiv v_{+}^{s} - v_{-}^{s}$
- linearization: $\tilde{U}/_{(t_s,t_{s+1})}(t) = U/_{(t_{s-1},t_s)}(t) \quad \forall t \in (t_s, t_{s+1})$

Time discontinuous Galerkin

- Time DG:

$$\int_{t_s}^{t_{s+1}} (U', v) + \varepsilon A_h(U, v) + b_h(\tilde{U}, v) dt + (\{U\}_s, v_+^s)$$
$$= \int_{t_s}^{t_{s+1}} \ell_h(v) dt \quad \forall v \in V_{h,\tau}^{p,q}, \ \forall s$$
$$(U_-^0, v) = (u^0, v) \quad \forall v \in V_{h,p,0}$$

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- error estimates derived for arbitrary order $q + 1$

$$\|e\|_{h,\tau,L^\infty(L^2)}^2 = O(h^{2p} + \tau^{2q+2}),$$
$$\|e\|_{h,\tau,L^2(H^1)}^2 = O(h^{2p} + \tau^{2q+2}),$$

Time discontinuous Galerkin summary

Advantages:

- favourable stability properties for arbitrary order

Time discontinuous Galerkin summary

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Disadvantages:

- not self–started
- very expensive

TDG solving

- we set orthonormal basis of $V_{h,p}$ as $\phi_1, \dots, \phi_N$ and orthonormal basis of $P^q(t_s, t_{s+1})$ as $\varphi_1, \dots, \varphi_{q+1}$

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-

$$A = (a_{i,j}) \qquad a_{ij} = A_h(\phi_j, \phi_i),$$
$$R = (r_{i,j}) \qquad r_{ij} = \int_{t_s}^{t_{s+1}} \varphi_i'(t)\varphi_j(t)\,\mathrm{d}t + \varphi_i(t_{m-1})\varphi_j(t_{m-1})$$

TDG solving

$$\begin{pmatrix} r_{1,1}I + \delta_{1,1}A & \cdots & r_{1,q+1}I + \delta_{1,q+1}A \\ \vdots & \ddots & \vdots \\ r_{q+1,1}I + \delta_{q+1,1}A & \cdots & r_{q+1,q+1}I + \delta_{q+1,q+1}A \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{q+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{q+1} \end{pmatrix}.$$

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$$AX + XR = B,$$

where $X,\ B \in \mathbf{R}^{N,q+1}$

TDG solving

Schur factorization

$$R = Z\,E\,Z^T$$

TDG solving

Schur factorization

$$R = Z\,E\,Z^T$$

$$AXZ + XZE = BZ$$

TDG solving

Schur factorization

$$R = Z\,E\,Z^T$$

$$AXZ + XZE = BZ$$

$$AY + YE = C$$

TDG solving

we evaluate vectors $y_k, \ k = 1, \dots, q+1$ sequentially

$$(A + e_{kk} I) y_k = c_k - \sum_{i=1}^{k-1} e_{ik} y_i ,$$

TDG solving

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$$(A + e_{kk} I) y_k = c_k - \sum_{i=1}^{k-1} e_{ik} y_i,$$

or

$$\left\{ \begin{array}{l} (A + e_{kk} I) y_k + e_{k\,k+1} y_{k+1} = c_k - \sum_{i=1}^{k-1} e_{ik} y_i \\ e_{k+1\,k} y_k + (A + e_{kk} I) y_{k+1} = c_{k+1} - \sum_{i=1}^{k-1} e_{i\,k+1} y_i \end{array} \right\},$$