A COMPARISON OF SOME A POSTERIORI ERROR ESTIMATES FOR FOURTH ORDER PROBLEMS

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INTRODUCTION

- a priori error estimates
- a posteriori error estimates
  - effectivity index
  - asymptotical exactness
- computational error estimates

- *h*-adaptive FEM
- *p*-adaptive FEM
- *hp*-adaptive FEM
- goal-oriented *hp*-adaptive FEM

I. Babuška, W. C. Rheinboldt: A posteriori error estimates for the finite element method. Internat. J. Numer. Methods Engrg. 12 (1978), 1597–1615.
I. Babuška, W. C. Rheinboldt: Error estimates for adaptive finite element computations. SIAM J. Numer. Anal. 15 (1978), 736–754.

AUTOMATIC *hp*-ADAPTIVITY

1. Generate the initial mesh $\mathcal{T}_h$.
2. Solve the problem on $\mathcal{T}_h$.
3. Compute the (global) error estimate. If the estimate is below the tolerance given, stop.
4. Compute an (analytical or computational) error indicator η_T for every element $T \in \mathcal{T}_h$.
5. Construct a new mesh $\mathcal{T}_h$ by refining some elements and by increasing polynomial degrees of some elements in the parts of the domain with the largest error indicated.
6. Go to Step 2.

AUTOMATIC *hp*-ADAPTIVITY

L. Demkowicz: Computing with *hp*-Adaptive Finite Elements. Vol. 1, 2. Chapman & Hall/CRC, Boca Raton, FL, 2007, 2008.

C. Schwab: *p*- and *hp*-Finite Element Methods. Clarendon Press, Oxford, 1998.

P. Šolín, K. Segeth, I. Doležel: Higher-Order Finite Element Methods. Chapman & Hall/CRC, Boca Raton, FL, 2004.

SOME CLASSES OF ANALYTIC A POSTERIORI ERROR ESTIMATORS

- residual estimator
- implicit estimator (based on the solution of local problems)
- hierarchic basis (multilevel) estimator
- gradient recovery based estimator (the averaging of gradient)
- global estimator (independent of the way the approximate solution is computed)

A SIMPLE SETTING OF THE MODEL PROBLEM

BIHARMONIC EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

Kirchhoff model of the vertical displacement of the mid-surface of a clamped plate subject to bending

4th order elliptic equation

$$\Delta^2 u = f \quad \text{in} \quad \Omega,$$

$\Omega \subset R^2$ is a connected bounded polygonal domain with boundary Γ, the Dirichlet boundary conditions are

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma,$$

$f \in L_2(\Omega)$

LOCAL RESIDUAL ESTIMATES FOR THE MODEL PROBLEM

WEAK SOLUTION

weak solution $u \in H_0^2(\Omega)$ of the problem: the identity

$$\int_\Omega \Delta u \Delta v = \int_\Omega f v, \quad \text{i.e.} \quad \langle F(u), v \rangle = 0$$

to be satisfied for all test functions $v \in H_0^2(\Omega)$

APPROXIMATE SOLUTION

family of regular triangulations $\mathcal{T}_h$, $h > 0$

space $X_h \subset H_0^2(\Omega)$ of all continuous piecewise polynomial functions of degree at most $k \geq 1$ and corresponding to $\mathcal{T}_h$

approximation of f: construct

$$f_h = \sum_{T \in \mathcal{T}_h} \pi_{l,T} f,$$

where $\pi_{l,T}$ is the L_2 projection onto the space $P_l(T)$ of polynomials of a fixed degree at most $l \geq 0$ on T

approximate solution $u_h \in X_h$ of the problem: the identity

$$\int_\Omega \Delta u_h \Delta v_h = \int_\Omega f_h v_h, \quad \text{i.e.} \quad \langle F_h(u_h), v_h \rangle = 0$$

to be satisfied for all test functions $v_h \in X_h$

LOCAL RESIDUAL ERROR ESTIMATOR

local error of the approximation of f by f_h on the triangle $T \in \mathcal{T}_h$

$$\varepsilon_T = \|f - f_h\|_{0;T}$$

residual error estimator on the triangle $T \in \mathcal{T}_h$

$$\eta_{V,T} = \Bigg(h_T^4 \|\Delta^2 u_h - f_h\|_{0;T}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} \big(h_E \| [\Delta u_h]_E \|_{0;E}^2 + h_E^3 \| [n_E \nabla \Delta u_h]_E \|_{0;E}^2 \big) \Bigg)^{1/2}$$

R. Verfürth: A Review of A Posteriori Error Estimation and Adaptive Mesh Refinement Techniques. John Wiley, Chichester, and B. G. Teubner, Stuttgart, 1996.

Theorem. *Let u and u_h be the weak and the approximate solutions of the problem. Then there are positive constants $c_1, \dots, c_6$ that depend only on the ratio h_T/ρ_T and the integers k and l such that the estimates*

$$\|u - u_h\|_2 \leq c_1 \left(\sum_{T \in \mathcal{T}_h} \eta_{V,T}^2 \right)^{1/2} + c_2 \left(\sum_{T \in \mathcal{T}_h} h_T^4 \varepsilon_T^2 \right)^{1/2} + c_3 \|F(u_h) - F_h(u_h)\| + c_4 \|F_h(u_h)\|$$

and

$$\eta_{V,T} \leq c_5 \|u - u_h\|_{2;\omega_T} + c_6 \left(\sum_{T' \subset \omega_T} h_{T'}^4 \varepsilon_{T'}^2 \right)^{1/2}$$

for all $T \in \mathcal{T}_h$ hold. The computable quantities $\|F(u_h) - F_h(u_h)\|$ and $\|F_h(u_h)\|$ represent the consistency error of the discretization and the residual of the discrete problem.

THE SAME SIMPLE SETTING OF THE MODEL PROBLEM

BIHARMONIC EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

4th order elliptic equation

$$\Delta^2 u = f \quad \text{in} \quad \Omega,$$

$\Omega \subset R^2$ is a connected bounded polygonal domain with boundary Γ, the Dirichlet boundary conditions are

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma,$$

$f \in H^{-1}(\Omega)$

LOCAL RESIDUAL ESTIMATES FOR THE MIXED FORMULATION OF THE PROBLEM

WEAK SOLUTION
$\{w = \Delta u, u\} \in X \times V,\ V = H_0^1(\Omega)$ and $X = H^1(\Omega)$

continuous bilinear forms

$$a(w, z) = \int_\Omega wz \text{ on } X \times X \quad \text{and} \quad b(z, u) = \int_\Omega \nabla z \cdot \nabla u \text{ on } X \times V$$

with scalar-valued functions u, w, and z

Ciarlet-Raviart weak formulation: Find $\{w, u\} \in X \times V$ such that

$$a(w, z) + b(z, u) = 0 \quad \text{for all} \quad z \in X,$$
$$b(w, v) + \int_\Omega fv = 0 \quad \text{for all} \quad v \in V$$

APPROXIMATE SOLUTION

family of uniformly regular triangulations $\mathcal{T}_h$, $h > 0$

second order approximation in $P_2(T)$, the space of polynomials of the second degree on T

finite element spaces

$$X_h = \{x_h \in X \mid x_h|_T \in P_2(T) \quad \text{for all} \quad T \in \mathcal{T}_h\},$$
$$V_h = \{v_h \in V \mid v_h|_T \in P_2(T) \quad \text{for all} \quad T \in \mathcal{T}_h\}$$

discrete formulation of the problem: Find $\{w_h, u_h\} \in X_h \times V_h$ such that

$$a(w_h, z_h) + b(z_h, u_h) = 0 \quad \text{for all} \quad z_h \in X_h,$$
$$b(w_h, v_h) + \int_\Omega f v_h = 0 \quad \text{for all} \quad v_h \in V_h$$

LOCAL RESIDUAL ERROR ESTIMATORS

put $f_h = \pi_{0,T} f$ and $\varepsilon_T = \|f - f_h\|_{0;T}$ on each triangle $T \in \mathcal{T}_h$

local residuals

$$\mathcal{P}_T(u_h) = -\Delta u_h + w_h, \qquad \mathcal{R}_T(w_h) = -\Delta w_h + f_h,$$
$$\mathcal{P}_E(u_h) = \left[\frac{\partial u_h}{\partial n}\right]_E, \qquad \mathcal{R}_E(w_h) = \left[\frac{\partial w_h}{\partial n}\right]_E$$

local residual error estimators

$$\eta^2_{C,T} = |T| \|\mathcal{P}_h(u_h)\|^2_{0;T} + \frac{1}{2} \sum_{E \in \mathcal{E}(T)} h_E \|\mathcal{P}_E(u_h)\|^2_{0;E},$$
$$\widetilde{\eta}^2_{C,T} = |T| \|\mathcal{R}_h(u_h)\|^2_{0;T} + \frac{1}{2} \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} h_E \|\mathcal{P}_E(u_h)\|^2_{0;E} + |T| \varepsilon^2_T$$

errors

$$e_h(u) = u - u_h, \qquad e_h(w) = w - w_h$$

Theorem. *Let $\{w, u\} \in X \times V$ and $\{w_h, u_h\} \in X_h \times V_h$ be the weak and the approximate solutions of the problem. Then there are positive constants C_1 and C_2 independent of h such that the estimates*

$$\|e_h(u)\|_1 + h\|e_h(w)\|_0 \leq C_1 \left(\left(\sum_{T \in \mathcal{T}_h} \eta_{C,T}^2 \right)^{1/2} + h^2 \left(\sum_{T \in \mathcal{T}_h} \widetilde{\eta}_{C,T}^2 \right)^{1/2} \right)$$

and

$$\eta_{C,T} + h^2 \widetilde{\eta}_{C,T} \leq C_2 \left(|e_h(u)|_{1;\omega_T} + h_T \|e_h(w)\|_{0;\omega_T} + h_T^3 \sum_{T' \subset \omega_T} \varepsilon_{T'} \right)$$

for all $T \in \mathcal{T}_h$ hold.

A. Charbonneau, K. Dossou, R. Pierre: A residual-based a posteriori error estimator for the Ciarlet-Raviart formulation of the first biharmonic problem. Numer. Methods Partial Differential Equations 13 (1997), 93–111.

A MORE GENERAL SETTING OF THE MODEL PROBLEM

FOURTH ORDER EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

$$\mathrm{div}^2 \Lambda(x, \mathrm{D}^2 u) = f \quad \text{in} \quad \Omega,$$

$\mathrm{D}^2 u$ is the Hessian matrix of a function $u : \Omega \to R$, $u \in H^2(\Omega)$, $\Lambda = [\lambda_{ik}]$, $\Lambda : \Omega \times R^{n\times n} \to R^{n\times n}$ is a matrix-valued function measurable and bounded with respect to the variable $x \in \Omega$ and of class C_2 with respect to the matrix variable $\Theta \in R^{n\times n}$, the domain $\Omega \subset R^n$ has a piecewise C_1 boundary Γ, and the Dirichlet boundary conditions are

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma,$$

$f \in L_2(\Omega)$

GLOBAL ESTIMATES FOR THE APPROXIMATE SOLUTION OBTAINED IN AN ARBITRARY WAY

WEAK SOLUTION

assume that the Jacobian arrays

$$\Lambda'(x,\Theta) = \frac{\partial \Lambda(x,\Theta)}{\partial \Theta} = \left[\frac{\partial \lambda_{rs}(x,\Theta)}{\partial \vartheta_{ik}}\right]_{i,k,r,s=1}^{n} \in R^{(n\times n)^2}$$

(tensor of order 4) are symmetric and positive definite, i.e. that there are constants $0 < m \leq M$ such that

$$m\|\Phi\|_{\mathsf{F}}^2 \leq (\Lambda'(x,\Theta)\Phi)\odot\Phi \leq M\|\Phi\|_{\mathsf{F}}^2 \quad \text{for all} \quad x\in\Omega, \quad \Theta,\Phi\in R^{n\times n}.$$

with $\odot$ denoting the double-dot (entry by entry) matrix product and F the Frobenius matrix norm

assume further that the mapping $\Lambda' : \Omega \times R^{n\times n} \to R^{(n\times n)^2}$ is Lipschitz continuous in the matrix variable $\Theta \in R^{n\times n}$ with a Lipschitz constant L

WEAK SOLUTION CONTINUED

a counterpart of the Friedrichs inequality is

$$\|w\|_0 \leq C_\Omega \|\mathrm{D}^2 w\|_0$$

for all $w \in H_0^2(\Omega)$ and a suitable constant $C_\Omega > 0$

weak solution $u \in H_0^2(\Omega)$ of the problem: the identity

$$\int_\Omega \Lambda(x, \mathrm{D}^2 u) \odot \mathrm{D}^2 v - \int_\Omega fv = 0$$

to be satisfied for all test functions $v \in H_0^2(\Omega)$

APPROXIMATE SOLUTION

any function $\bar{u} \in H_0^1(\Omega)$ can be considered to be the approximate solution (the way it has been computed is not of interest)

measure of the error

$$
\begin{aligned}
E(\bar{u}) &= \int_\Omega (\Lambda(x, \mathrm{D}^2\bar{u}) - \Lambda(x, \mathrm{D}^2 u)) \odot (\mathrm{D}^2\bar{u} - \mathrm{D}^2 u) \\
&= \int_\Omega \Lambda(x, \mathrm{D}^2\bar{u}) \odot (\mathrm{D}^2\bar{u} - \mathrm{D}^2 u) - \int_\Omega f(\bar{u} - u)
\end{aligned}
$$

GLOBAL ERROR ESTIMATORS

$$\eta_K(\Psi, w, \bar{u}) = \Big(m^{-1/2} C_\Omega \|\mathrm{div}^2\Psi - f\|_0 + \tfrac{1}{2} L m^{-3/2} \delta(\Psi, w, \bar{u}) + \big((\Lambda(x, \mathrm{D}^2\bar{u}) - \Psi, \mathrm{D}^2\bar{u} - \Lambda^{-1}(x, \Psi)\big)_0 + \tfrac{1}{2} L m^{-1} \delta(\Psi, w, \bar{u}) \|\mathrm{D}^2\bar{u} - \Lambda^{-1}(x, \Psi)\|_0 \big)^{1/2} \Big)^2,$$

$$\delta(\Psi, w, \bar{u}) = (M\|\Lambda^{-1}(x, \Psi) - \mathrm{D}^2 w\|_0 + C_\Omega \|\mathrm{div}^2\Psi - f\|_0) \times \|\mathrm{D}^2\bar{u} - \Lambda^{-1}(x, \Psi)\|_\infty,$$

$\Psi \in H(\mathrm{div}^2, \Omega) \cap L_\infty(\Omega, R^{n\times n})$ is an arbitrary matrix-valued and $w \in H_0^2(\Omega)$ an arbitrary scalar-valued function, m and M are the positive definiteness constants, C_Ω the Friedrichs inequality constant, and L the Lipschitz continuity constant of Λ'

another, computationally more friendly estimator $\widetilde{\eta}_K(\Psi, w, \bar{u})$

Theorem. *Let* $u \in H_0^2(\Omega)$ *be the weak solution of the problem and* $\bar{u} \in W^{2,\infty}(\Omega)$ *an arbitrary function. Then*

$$E(\bar{u}) \leq \eta_K(\Psi, w, \bar{u})$$

for any $\Psi \in H(div^2, \Omega) \cap L_\infty(\Omega, R^{n\times n})$ *and* $w \in H_0^2(\Omega)$.

the same statement holds for $\widetilde{\eta}_K(\Psi, w, \bar{u})$ with some other class of matrix-valued functions Ψ

Theorem. *If the weak solution* $u \in W^{2,\infty}(\Omega)$ *then the estimator* η_K *is sharp, i.e.*

$$\min_{\Psi \in H(div^2,\Omega)\cap L_\infty(\Omega,R^{n\times n}),\ w\in H_0^2} \eta_K(\Psi, w, \bar{u}) = E(\bar{u}).$$

J. Karátson, S. Korotov: Sharp upper global a posteriori error estimates for nonlinear elliptic variational problems. Appl. Math. 54 (2009), 297–336.

PROPERTIES OF ANALYTICAL A POSTERIORI ERROR ESTIMATORS

ADVANTAGES

- computed from the approximate solution
- fast/cheap computation
- asymptotical exactness, often good behavior even for finite h

DRAWBACKS

- usually depend on unknown constants or functions
- usually constructed only for lowest-order polynomial approximation

PROPERTIES OF COMPUTATIONAL ERROR ESTIMATORS (REFERENCE SOLUTIONS)

- standard for ODE's
- quite universal, but more expensive: if $u_{h,p}$ is the approximation on the current mesh then $u_{\mathrm{ref}} = u_{h/2,p+1}$
- provide more complex information on the behavior of the error
- used in multilevel solution procedures
- used in adaptive (in particular *hp*-adaptive and goal-oriented) algorithms