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INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME-VARYING DELAY

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Abstract

Integro-differential equations with time-varying delay can provide us with realistic models of many real world phenomena. Delayed Lotka-Volterra predator-prey systems arise in ecology. We investigate the numerical solution of a system of two integro-differential equations with time-varying delay and the given initial function. We will present an approach based on q-step methods using quadrature formulas.

1. Introduction

Integro-differential equations (IEs) are one of the most important mathematical tools used in modelling problems of many real world phenomena. Here, we consider the Lotka-Volterra like predator-prey model [1]. This system of two IEs is frequently used to describe the dynamics of biological systems in which two species interact. One is the population of predators of the size $x_1(t)$ and the other is that of preys of the size $x_2(t)$

$$x_1'(t) = \left[c - k_1 x_2(t) - \int_{-\tau}^{0} \alpha_1(x_2(t+s))ds\right] x_1(t)$$

$$x_2'(t) = \left[-c + k_2 x_1(t) - \int_{-\tau}^{0} \alpha_2(x_1(t+s))ds\right] x_2(t)$$

where $x_1'(t)$ and $x_2'(t)$ represent the growth of the two populations with time, c, k_i, α_i are parameters representing the interaction of the two species.

Also, one of the models for human immunodeficiency virus (HIV) in a homogeneously mixed single-gender group with distributed waiting times can be described using IEs, see [3].

So elaboration of numerical methods for IEs is a very important problem. Presently, various specific numerical methods are constructed for solving specific IEs. Most investigations are devoted to numerical methods for systems with discrete delays, see e.g. [2].

The approach described in this article has been applied to numerical solution of integro-differential equations with time-varying delay (IDETVD) .

2. Equations with time-varying delay

Delay differential equations (DDEs) represent the principal form of mathematical models occuring in Ecology. In DDEs, also called functional differential equations or time-delay systems, dependent variables are simultaneously evaluated at more than one value of the independent variable.

The considered DDE Cauchy problem is

$$\begin{aligned} x' &= f(t, x(t+\tau_1), \cdots, x(t+\tau_k)), \quad t \geq t_0, \\ x(t) &= \Psi(t), \quad t \leq t_0 \end{aligned}$$

f is a function with the independent variable t representing time, dependent variable $x(t)$ is a phase vector and $x(t+\tau_j)$, $\tau_j \in < -r_j, 0 >$, $\quad j = 1, 2, \cdots, k$ are the functions characterizing the influence of the pre-history of the phase vector on the dynamics of the system. A class of DDE with constant delay $\tau_j, j = 1, 2, \cdots, k$ is called DDEs with discrete delay. Supposed that delay $\tau_j = \tau_j(t)$ we speak about differential equations with time-varying delay.

Let us consider some of them. The delay logistic equation

$$x'(t) = r(t)x(t)\left(1 - \frac{x(\tau(t))}{K}\right), \quad \tau(t) \leq t$$

describes a delay population model and is known as Hutchinson's equation [2]. One can see that it is insufficient to know the initial value only to define the phase vector $x(t)$. It is also necessary to know an inital function (initial pre-history) $\Psi(t)$. Hence the DDEs are generalizations of the ODEs such that the velocity $x'(t)$ of a process depends also on the pre-history $x(\tau(t))$, $\tau(t) \leq t$.

Delay can also be distributed as in the equation

$$x'(t) = f\left(t, x(t), \int_{\tau(t)}^{0} \alpha(t, s, x(t+s))ds\right).$$

So, the Volterra integro-differential equations

$$x'(t) = f\left(t, x(t), \int_{0}^{t} \beta(t, s, x(s))ds\right)$$

represent a special class of DDEs with distributed delays.

The purpose of this article is to derive a numerical method for the approximate solution of delay differential systems with time-varying delay of the form

$$x'(t) = f\left(t, x(t), x(\tau_1(t)), \int_{\tau(t)}^{0} \chi(t, s, x(t+s))ds\right).$$

In [3], Kim and Pimenov proposed an exact solution to a system of IDETVD

$$x_1'(t) = -\sin(t)x_1(t) + x_1(t - \frac{t}{2}) - \int_{-t/2}^{0} \sin(t+s)x_1(t+s)ds - \mathrm{e}^{\cos(t)} \tag{1}$$

$$x_2'(t) = -\cos(t)x_2(t) + x_2(t - \frac{t}{2}) - \int_{-t/2}^{0} \cos(t+s)x_2(t+s)ds - \mathrm{e}^{\sin(t)} \tag{2}$$

corresponding to an inital function

$$\begin{array}{rcl} \Psi_1(s) & = & \mathrm{e}^{\cos(s)} \\ \Psi_2(s) & = & \mathrm{e}^{\sin(s)} \end{array}, \quad s \leq 0.$$

The solution $(x_1(t), x_2(t))^T, t \in [0, \infty)$ of (1), (2) has the form

$$\begin{array}{rcl} x_1(t) & = & \mathrm{e}^{\cos(t)}, \\ x_2(t) & = & \mathrm{e}^{\sin(t)}. \end{array}$$

Then by considering the maximum absolute errors in the solution at grid points for different choices of step size, we can conclude how further presented approaches produce accurate results in comparison with those exact ones.

3. A numerical approach

The most popular numerical approaches for solving Cauchy problem of ODEs are called finite difference methods. Approximate values are obtained for the solution at a set of grid points $\{t_n : n = 1, 2, \cdots, N\}$ and the approximate value at each point t_{n+1} is obtained by using some of values obtained in previous steps. The best known methods are Euler's methods (explicit, implicit), trapezoidal method, Milne's methods, Adams methods.

Most integrals cannot be evaluated explicitly and with many others it is often faster to integrate them numerically rather than evaluating them exactly. Formulas using such interpolation with evenly spaced grid points are the composite trapezoidal rule and the composite Simpson's rule. These Newton-Cotes formulas can be used to construct a composite method with mentioned methods.

The simplest way how to solve our problem is the combination of the explicit Euler's method with the trapezoidal rule, outlined in the following procedure solving the problem (1) on an equidistant mesh $t_{n+1} - t_n = h$, where we abbreviate $x_1(t)$ by $x(t)$.

First, the trapezoidal rule is defined by applying

$$x(t_{n+1}) = x(t_n) + \frac{h}{2}\Bigg[-\sin(t_n)x(t_n) + x(t_n/2) - \int\limits_{-t_n/2}^{0} \sin(t_n+s)x(t_n+s)ds - \mathrm{e}^{\cos(t_n)}$$

$$- \sin(t_{n+1})x(t_{n+1}) + x(t_{n+1}/2) - \int\limits_{-t_{n+1}/2}^{0} \sin(t_{n+1}+s)x(t_{n+1}+s)ds - \mathrm{e}^{\cos(t_{n+1})}\Bigg]$$

to successive subintervals $[t_n, t_{n+1}]$, where

$$h = 2H,\ t_n = 2kH,\ t_{n+1} = 2(k+1)H,\ k = 0, 1, 2, \ldots.$$

Hence,

$$x(2(k+1)H) = x(2kH) + H\Bigg[-\sin(2kH)x(2kH) + x(kH)$$

$$-\int_{-kH}^{0} \sin(2kH+s)x(2kH+s)ds - e^{\cos(2kH)} - \sin(2(k+1)H)x(2(k+1)H)$$

$$+ x((k+1)H) - \int_{-(k+1)H}^{0} \sin(2(k+1)H+s)x(2(k+1)H+s)ds - e^{\cos(2(k+1)H)}\Bigg]$$

Since

$$\int_{-(k+1)H}^{0} A = \int_{-(k+1)H}^{-kH} A + \int_{-kH}^{0} A$$

and letting

$$I(k) = \int_{-kH}^{0} \sin(2kH+s)x(2kH+s)ds$$

$$I(k+1) = \int_{-(k+1)H}^{0} \sin(2(k+1)H+s)x(2(k+1)H+s)ds$$

we have

$$x(2(k+1)H) = x(2kH) + H\big[-\sin(2kH)x(2kH) + x(kH) - I(k) - e^{\cos(2kH)} - \\ -\sin(2(k+1)H)x(2(k+1)H) + x((k+1)H) - I(k+1) - e^{\cos(2(k+1)H)}\big]$$

Now, we shall confine our discussion to evaluating $I(k)$ and $I(k+1)$ approximately. For a sufficiently small mesh size H the composite trapezoidal rule gives a good approximation to the integrals

$$I(k) = \sum_{p=0}^{k-1} \frac{H}{2}\big[\sin((k+p)H)x((k+p)H) + \sin((k+p+1)H)x((k+p+1)H)\big]$$

$$I(k+1) = \sum_{p=0}^{k} \frac{H}{2}\big[\sin((k+p+1)H)x((k+p+1)H) \\ + \sin((k+p+2)H)x((k+p+2)H)\big]$$

However, it is possible to obtain finite sums which give better approximations by the same amount of computation. One sees, immediately, that $x(t_{n+1})$ can be computed when t_{n+1} is the even multiple of H. If t_{n+1} is the odd multiple of H then we apply explicit Euler method to the model equation on an equidistant mesh $t_{n+1} - t_n = h$.

Then, the explicit Euler method is defined by applying

$$x(t_{n+1}) = x(t_n) + h\left[-\sin(t_n)x(t_n) + x(t_n/2) - \int_{-t_n/2}^{0} \sin(t_n+s)x(t_n+s)ds - e^{\cos(t_n)}\right]$$

to successive subintervals $[t_n, t_{n+1}]$, where $h = H, \quad t_n = 2kH, \quad t_{n+1} = (2k+1)H$, $k = 0, 1, 2, \cdots$. This yields

$$x((2k+1)H) = x(2kH) + H\Bigg[-\sin(2kH)x(2kH) + x(kH) - \int_{-kH}^{0} \sin(2kH+s)x(2kH+s)ds - \mathrm{e}^{\cos(2kH)}\Bigg]$$

It can be seen that this formula contains the integral $I(k)$.

Also,

$$x((2k+1)H) = x(2kH) + H\left[-\sin(2kH)x(2kH) + x(kH) - I(k) - \mathrm{e}^{\cos(2kH)}\right].$$

4. Numerical experiments

In order to test the viability of the proposed composite methods and to demonstrate its convergence computationally we have considered several tests with some steps, to assess the convergence property and efficiency of methods mentioned in Section 3.

We divide the time interval $t \in [0, 6.3]$ into N subintervals in order to obtain the approximate values for the solution at the grid points t_n. Here we are only interested in showing the errors of the solution at some grid points. The idea was to calculate the numerical solution by Milne-Simpson's method of 5-th order with the Simpson's rule on an equidistant mesh $t_{n+1} - t_n = h = 0.003$. Table 1 contains the errors in this numerical solution in selected gridpoints.

Numerical and exact results are illustrated in Figure 1 in the time varying plane and in Figure 2 in the phase plane also.

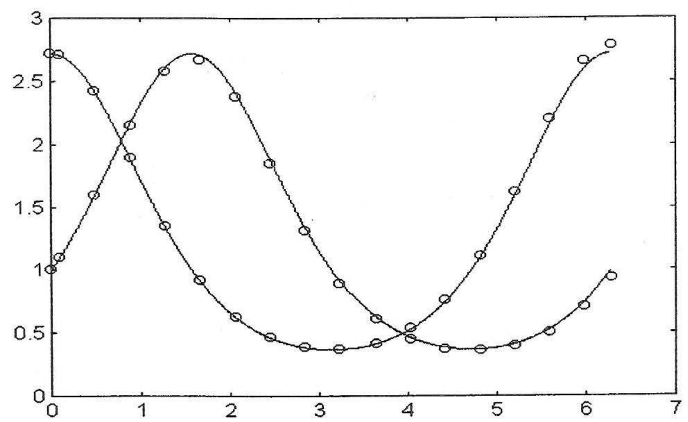

Figure 1: Graph of $x_1(t)$ and $x_2(t)$ versus time.

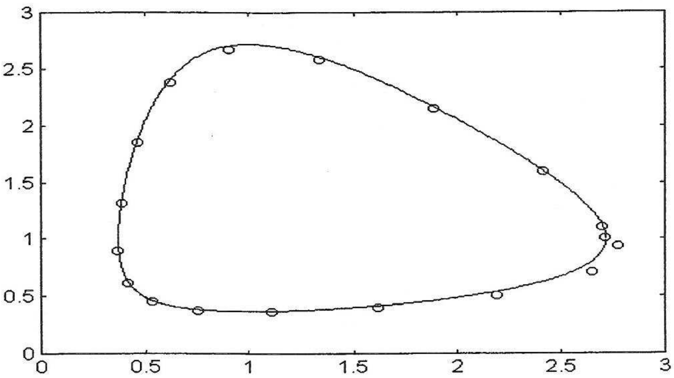

Figure 2: Graph of $x_1(t)$ versus $x_2(t)$.

t	$x_1(t)$	error of $x_1(t)$	$x_2(t)$	error of $x_1(t)$
2.1	0.6035988	0.0052716	2.3707579	0.0185469
4.2	0.6124669	0.0043827	0.4182918	0.0038792
6.3	2.7178976	0.0118452	1.0169558	0.0298354

Table 1: Errors in the numerical solution.

The solid lines indicate the graphs of exact solution $(x_1(t), x_2(t))^T$ with $x_1(0) = \mathrm{e}$, $x_2(0) = 1$, $t \in [0, 6.3]$. Our program begins with the second order trapezoidal formula and the explicit Euler's formula, the accuracy then increases as extra starting values become available.

References

[1] Gopalsamy, K.: *Stability and oscillation in delay differential equations of population dynamics.* Kluwer Academic Publishers, Dordrecht, Boston, London, 1992.

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