

## DRIVER'S INFLUENCE ON KINEMATICS OF ARTICULATED BUS REAR AXLE

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### Abstract

This paper studies kinematic properties of the rear axle of the particle coach as function of driver's activity. The main goals are the prediction of the trajectory, the computation of the vector of velocity of each wheel of the rear axle as a function of the real velocity vector of the front coach axle and the real curvature of the bus trajectory. The computer algebra system MAPLE was used for all necessary computations.

## 1. Introduction

### 1.1. Classical problem of kinematic

In following computations we should use these main variables:  $X(t), Y(t)$  – General coordinates of the moving body, later coordinates of the midpoint of the central axle of the articulated bus, which is equal to the joint point of the rear – towed axle.  $x(t), y(t)$  – Coordinates of the midpoint of the towed axle.  $L$  – Constant distance between midpoints of the central axle - joint, and rear - towed axle, see Figure 1.

The classical problem of kinematics is the computation of the speed  $\vec{V}(t)$  and the acceleration vector  $\vec{A}(t)$  of a body as a function of time when the location of the body is given by the functions  $\vec{P}(t) = [X(t), Y(t)]$ . The next step is the computation of the tangential acceleration  $A_t(t)$ , which changes the absolute value of the velocity and the normal acceleration  $A_n(t)$ , which changes the direction of the velocity. And finally, the function of the center of the osculation circle of the trajectory  $\vec{C}(t)$  and its radius  $R(t)$  are derived. These functions can also be found in [4, 2, 9].

### 1.2. The influence of the driver

The driver controls the bus using the gas and the brake pedal – he controls the absolute value of the velocity of the bus  $|\vec{V}(t)|$ . Furthermore – using the steering wheel – he controls the radius of the osculating circle  $R(t)$ , on which the bus is currently moving. For further calculations it is useful to use the inverse value of the radius of the osculation circle – the curvature of trajectory  $k(t) = R(t)^{-1}$ . By combining these two controls the bus driver keeps the bus moving smoothly on the road.

## 2. Inverse problem

Let us assume that we know the temporal behavior of driver's operations. Thus we know the functions of the speed magnitude  $|\vec{V}(t)| = v(t)$  and curvature  $k(t)$ . Then the problem is to compute the trajectory of the bus and the related kinematics variables. For this we need to solve a non-linear system of two ordinary differential equations of second and first order, they are solved in [1, 5]

$$\sqrt{\dot{X}^2 + \dot{Y}^2} = v(t), \quad \frac{\ddot{Y}\dot{X} - \ddot{X}\dot{Y}}{(\dot{X}^2 + \dot{Y}^2)^{\frac{3}{2}}} = k(t). \quad (1)$$

After some algebraic manipulations the equations (1) are transformed to an explicit system of two differential equations of order two:

$$\ddot{X} = \frac{-\dot{Y} k(t) v(t)^2 + \frac{dv(t)}{dt} \dot{X}}{v(t)}, \quad \ddot{Y} = \frac{\dot{X} k(t) v(t)^2 + \frac{dv(t)}{dt} \dot{Y}}{v(t)}. \quad (2)$$

Given an initial velocity  $v_0 = |\vec{V}(0)|$  and its initial direction defined by the angle  $\phi_0$  and the initial position of the bus  $[X_0, Y_0]$ , the solution of (2) can be found to be (see [4,5])

$$X = \int_0^t v(\tau) \cos(f) d\tau + X_0, \quad Y = \int_0^t v(\tau) \sin(f) d\tau + Y_0, \quad (3)$$

where  $f = \phi_0 + \int_0^\tau v(\tau) k(\tau) d\tau$ . This is an analytic solution, however, even for simple functions  $v(t)$  and  $k(t)$  it will not be possible to compute explicit expressions for the integrals. A considerable advantage of this result is that it allows to numerically integrate the position for any given time  $t$ . We have not to be concerned with accumulation of rounding errors as e.g by integrating the system (2) with some numerical methods, like Runge-Kutta, see [8].

## 3. Generalized tractrix as model of the trajectory of the rear axle

Let us assume that the joint of the articulated bus is located in the middle of second axle and that the trajectory of the joint is given by  $[X, Y]$ . The centre of the rear axle, given by  $[x, y]$  - the towed axle - is to be computed. The centres of both axles have to have a constant distance  $L$  and the velocity vector of the center of the towed axle has to pass the joint, see Figure 1.

From these conditions (see [3]) we obtain the system of differential equations  $[X, Y]$  and  $[x, y]$ .

$$\dot{x} = \frac{\Delta X (\Delta X \dot{X} + \Delta Y \dot{Y})}{L^2}, \quad \dot{y} = \frac{\Delta Y (\Delta X \dot{X} + \Delta Y \dot{Y})}{L^2}, \quad \text{where} \quad \begin{array}{l} \Delta X = x - X, \\ \Delta Y = y - Y. \end{array} \quad (4)$$

It is a system of two non-linear differential equations of first order, which for simple functions  $X$  and  $Y$  is relatively easy to solve.

But if we introduce for  $X$  and  $Y$  the expressions of Equations (3), we get a very complex system of differential equations, for which it is first necessary to solve for  $X$  and  $Y$  by the numerical integration. This combination of numerical integration and solving of differential equations is too complex for the computer algebra system MAPLE. It is not possible to use successfully direct numerical solution of equations using the command **dsolve** together with the parameter **numeric**.

### 3.1. Numerical integration of the equation of motion

It is possible to solve the system (3) together with Equations (4) numerically using Runge-Kuttas method, see [8]. We implemented this in MAPLE as procedure **RK45**. This procedure determines the position and velocity of the towed axles centre at time  $t + \Delta t$ . The next procedure, named **STEP**, see (5), defines the magnitude of time step  $\Delta t$  using a step size control. For the first iteration a random time step magnitude is chosen, e.g.  $\Delta t = 1$  and the position for this time is calculated. Time  $t$  and coordinates  $x, y$  are saved in the vector  $R1$ . Similarly the position is calculated in the same procedure, but in two steps with a half time step size  $\Delta t/2$  and saved as a vector  $R2$ . If the difference between these vectors is smaller than the required accuracy,  $|R1 - R2| \leq 10^{-6}$ , we add the resulting position, saved in the vector  $R1$  to vector  $\Lambda$ . Otherwise we reduce the size of time step by half and repeat the process. At the end of the iteration procedure the vector  $\Lambda$  will contain vectors – ordered triplets containing the time and the towed axles position coordinates of the each iteration step.

```
STEP := proc( $U$ ) local  $l, R2$ ; global  $R1, \Lambda, \Delta t, t$ ;
 $l := U[]$ ;  $R2 := [\mathbf{RK45}(\mathbf{RK45}(l, \frac{\Delta t}{2}), \frac{\Delta t}{2})]$ ;  $R1 := [\mathbf{RK45}(l, \Delta t)]$ ;
if sqrt(add( $u^2, u = R1 - R2$ ))  $\leq 10^{-6}$  then  $\Lambda := [\Lambda[], R1]$ ;  $t := t + \Delta t$ ; (5)
else  $\Delta t := \frac{\Delta t}{2}$  end if
end proc
```

## 4. Practical application

Let us take as example a passing of a rectangular turn when the bus is breaking. For this case we consider

$$v(t) = V_0 - at, \quad k(t) = \frac{4t(T_f - t)}{T_f^2 \rho}, \quad (6)$$

where  $V_0$  is the initial velocity,  $a$  is deceleration,  $T_f$  is the period of turn passing and  $\rho$  is the least diameter of a passed turn.

If we choose the direction in time  $t = 0$  parallel to  $x$  axis, therefore  $\phi_0 = 0$ , the turn will be finished at the moment, when the vector of immediate velocity  $[\dot{X}, \dot{Y}]$  will be parallel to  $y$ . Therefore it is stated that  $\dot{X} = 0$ . From this condition it is obvious, that because of Equation (3) we have (details can be found in [6, 7])

$$T_f = \frac{2V_0 - \sqrt{4V_0^2 - 6a\pi\rho}}{2a}. \quad (7)$$

#### 4.1. Numerical integration

As particular values we take  $V_0 = 10 \text{ m s}^{-1}$ ,  $a = 0.5 \text{ m s}^{-2}$ ,  $\rho = 20 \text{ m}$ ,  $x_0 = 0 \text{ m}$ ,  $y_0 = 0 \text{ m}$  and  $L = 4 \text{ m}$ . For these values the time necessary to pass the turn is  $T_f = 5.45681 \text{ s}$ . The initial time is  $t = 0 \text{ s}$  and for the initial time step we choose  $\Delta t = T_f$ . Now we create the  $\Lambda$  list, its first element will be  $[t, -L, 0]$ , then  $\Lambda := [[0, -4, 0]]$ . Procedure **STEP** determines the first step size of the time step as  $\Delta t = \frac{T_f}{128} = 0.04263 \text{ s}$  and then executes 128 integration steps. For specific integration times it is possible to compute using Equations (3) the position of middle axles centre point. Due to Figure 2 and the following relationship (8) it is hence possible to calculate the position of front, middle and rear – towed axle.

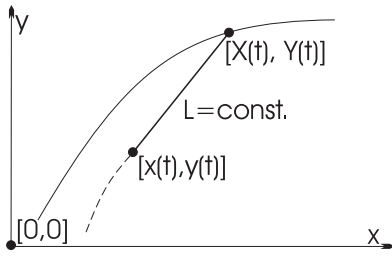


Figure 1: Deriving the motion equation of towed axle.

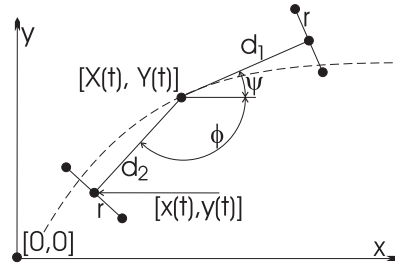


Figure 2: Calculation for single wheels of a bus.

$$\text{Wheel} = [X, Y] + d [\cos(\alpha), \sin(\alpha)] + r [-\sin(\alpha), \cos(\alpha)], \quad (8)$$

for  $d$  we can take  $d_1$  – the distance between centrepoints of middle and front axle, or  $d_2$  – the distance between middle and rear axles centrepoints. For  $\alpha$  we can take  $\psi$  – the directional vector pointing from the middle axle to the front axles midpoints. This is the directional vector of the velocity  $[\dot{X}, \dot{Y}]$  or  $\phi$  – the directional vector pointing from the joint of the middle point of the rear axle,  $r$  = wheel spacing of single axles. Angular sizes  $\phi$  and  $\psi$  could be easily solved using the vector calculation. The result of the calculation could therefore be a graph on Figure 3., depicting the trajectories of single wheels, or a graph on Figure 4, which represents the angle of cranking of the bus joint.

#### 5. Conclusion and discussion

We have developed a method which allows for any velocity function  $v(t)$  and trajectory curvature function  $k(t)$  to compute all important kinematic variables of the articulated bus. This concerns not only the wheels but can be applied for any arbitrary point inside the bus. Just take for that the appropriate sizes of variables  $d_1$ ,  $d_2$ , and  $r$  matching the Figure 2 and Equation (8). Furthermore, it is possible to determine the acceleration of any point, including the points which correspond to points of contact between the wheels and the road. This knowledge

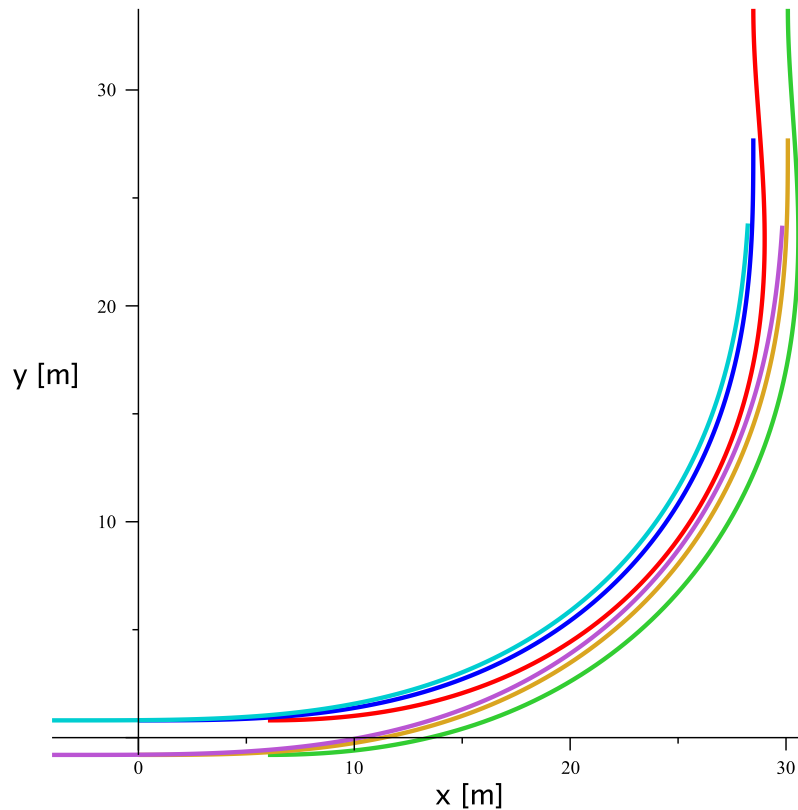


Figure 3: Trajectory of separate wheels of the bus.

of acceleration could be used for the determination of the adhesion threshold limits. This procedure could be also used for the inverse problem. From the moment of the adhesion loss to breakaway it is possible to experimentally find such a velocity and trajectory curvature functions, that caused the skid. Therefore it is possible from the trajectory – a braking track – to estimate the drivers actions, that preceded this event.

From the knowledge of acceleration inside the bus it is possible to perform the calculations of general force, affecting the whole bus as well as individual passengers. Knowledge of this general force is an important factor affecting the stability of the bus. The force affecting the single passenger is a limiting factor for their safety. The method mentioned above allows us to simulate the drivers behavior and the impact on safety of passengers due to their position inside the bus.

### Acknowledgments

The work has been supported by research and development projects IP 22/2012, financed by IGA AF MENDELU.

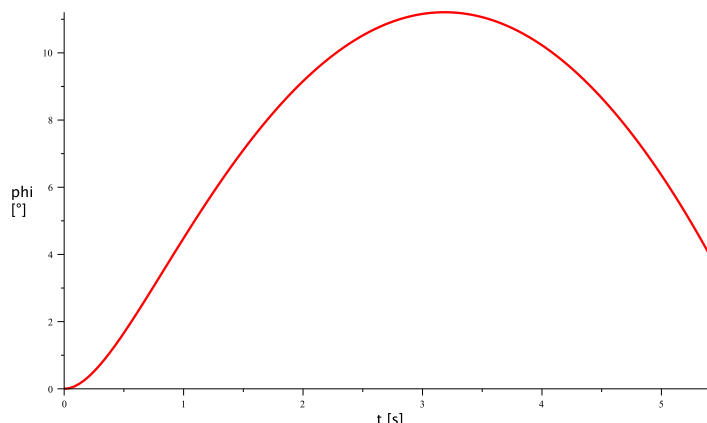


Figure 4: Angle of cranking the joint of the bus.

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