Dva přístupy k vynucení teploty při chlazení proudící vodou.

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Technical motivation

Shape optimization of the cooling plunger cavity

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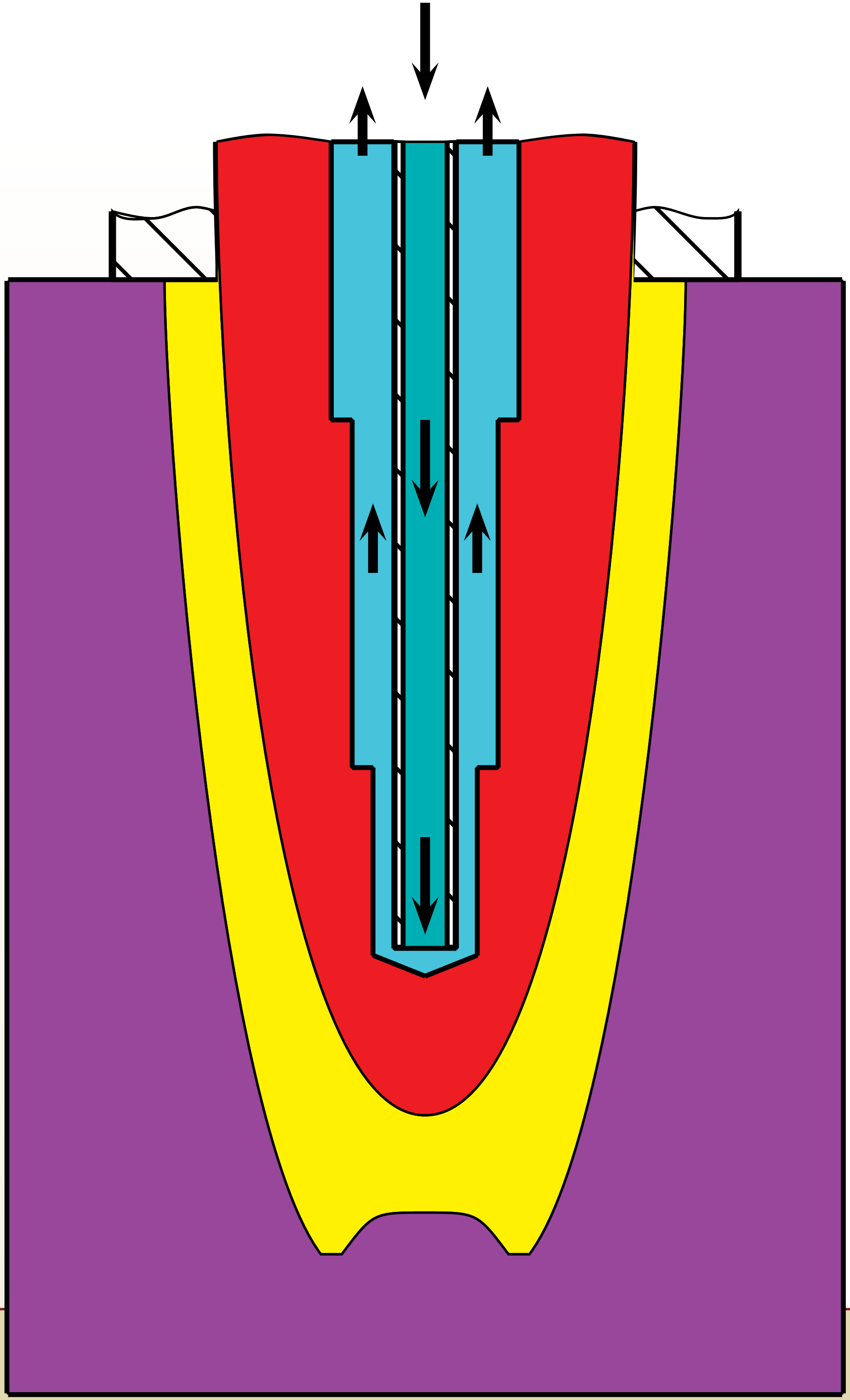

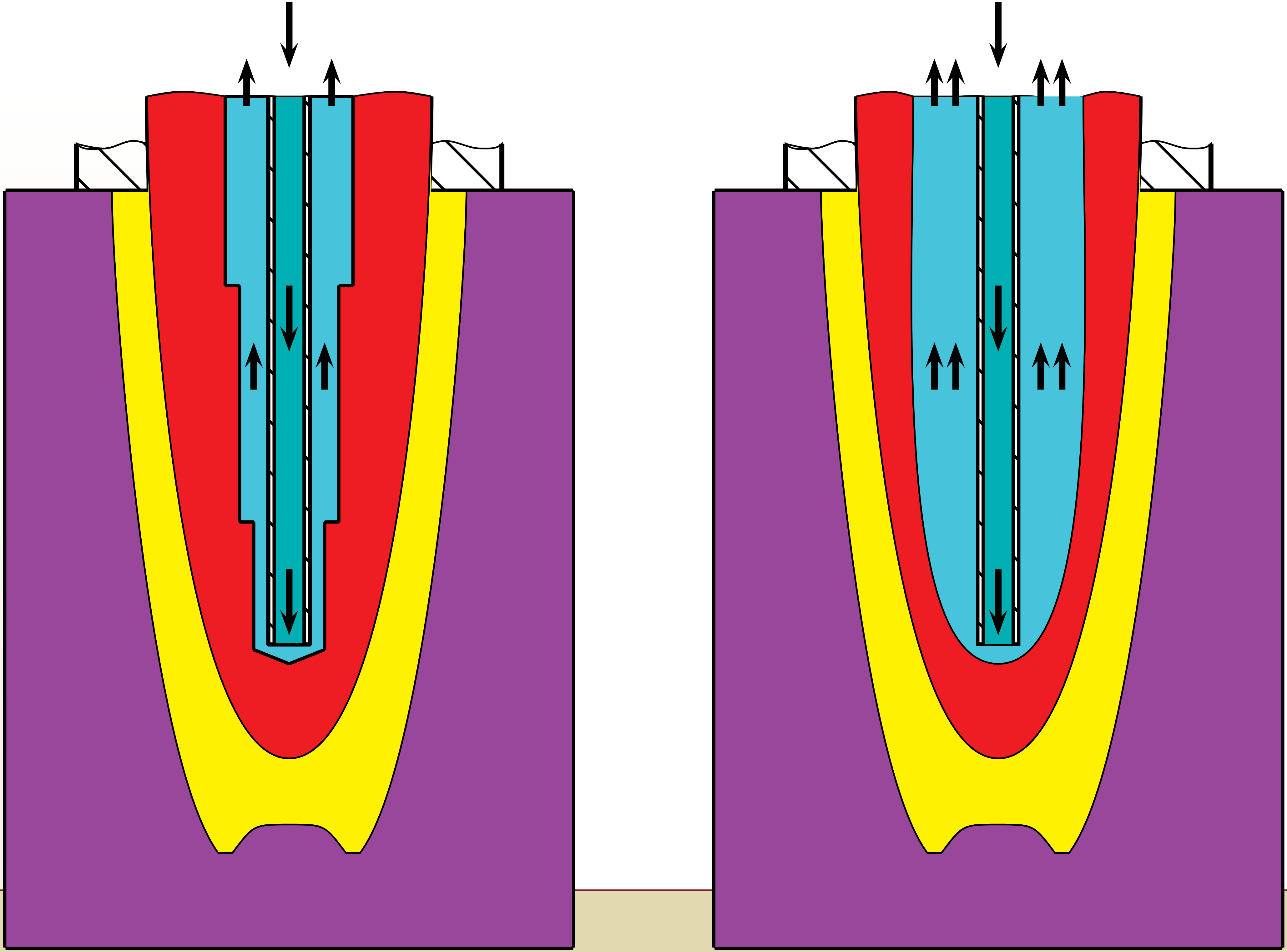

Shape optimization of the cooling plunger cavity

We denote

$$F_2^e(x) = \begin{cases} 0 & \text{for} \quad x \in [0,\, x_2^e] \\ f_2^e(x) & \text{for} \quad x \in [x_2^e,\, 1] \end{cases} \quad , \tag{1}$$

where $x_2^e \in [s_{\min},\, 1]$, ($s_{\min} > 0$ is fixed constant given by the minimal thickness of the plunger wall), $f_2^e \in C^{(0),1}([x_2^e,\, 1])$, $f_2^e(x_2^e) = 0$ and $0 < f_2^e(x) < f_1(x) - s_{\min}$, where f_1 is fixed given increasing function, which represents outward shape of the plunger. Further we assume that $a < f_2^e(x) - s_2$ for $x \in [x_3^e, 1]$, where $a > 0$ represents radius of supply tube and $s_2 > 0$ minimal admissible split width between the inner wall of plunger cavity and the supply tube, $x_3^e \in]x_2, 1]$ deepness of insertion of the tube.

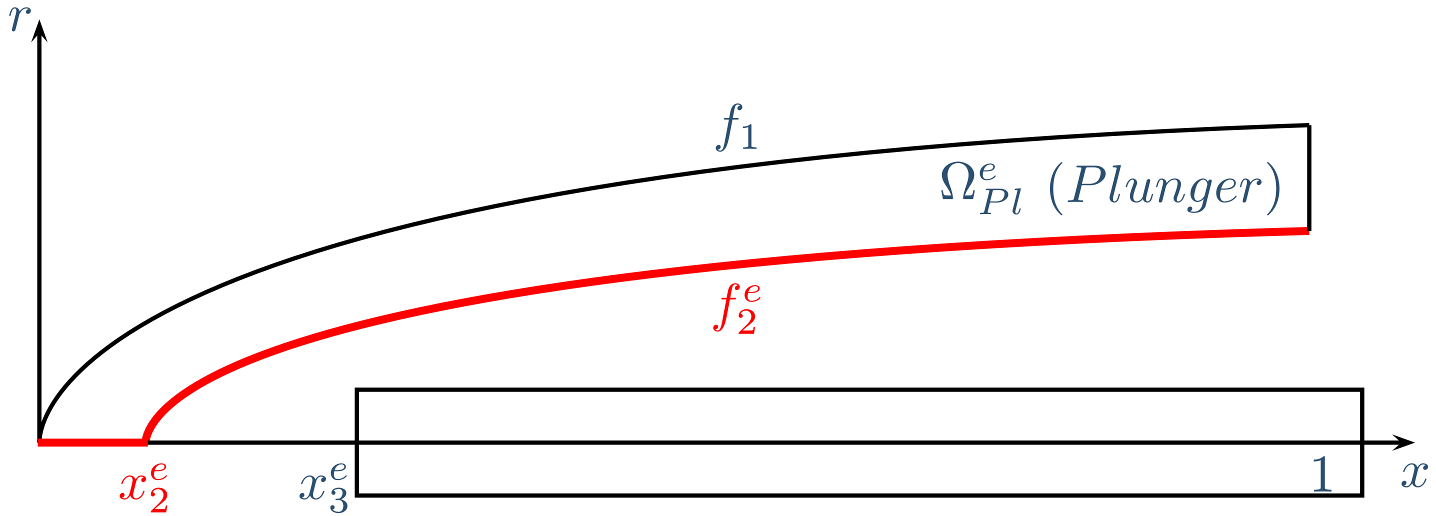

Fig 1

Shape optimization of the cooling plunger cavity

We denote **the set of admissible functions** as

$$U^e_{ad} = \{ \; F^e_2(x) \in C^{(0),1)}([0,1])\,; \; F^e_2(x) = \begin{cases} 0 & \text{for} \quad x \in [0,\, x^e_2] \\ f^e_2(x) & \text{for} \quad x \in [x^e_2,\, 1] \end{cases} \quad ,$$

$$x^e_2 \in [s_{\min},\, 1], \; s_{\min} > 0, \; f^e_2 \in C^{(0),1}([x^e_2,\, 1]), \; f^e_2(x^e_2) = 0,$$

$$0 < f^e_2(x) < f_1(x) - s_{\min}, \; f_1 \text{ given},$$

$$a < f^e_2(x) - s_2 \text{ for } x \in [x^e_3, 1], \; a > 0, \; s_2 > 0, \; x^e_3 \in]x_2, 1]\} \,,$$

where the function F^e_2 describes the technological constraint for inner cavity surface.

We assume the region Ω^e_{Pl}, which depends on the design function $F^e_2(x)$, and which is defined by the formula

$$\Omega^e_{Pl} = \{(x,\, r) \in R^2\,; \; F^e_2(x) < r < f_1(x), \quad \text{for} \quad x \in [0,\, 1]\} \;,$$

where f_1 is fixed given increasing function, which represents inner shape of plunger.

Shape optimization of the cooling plunger cavity

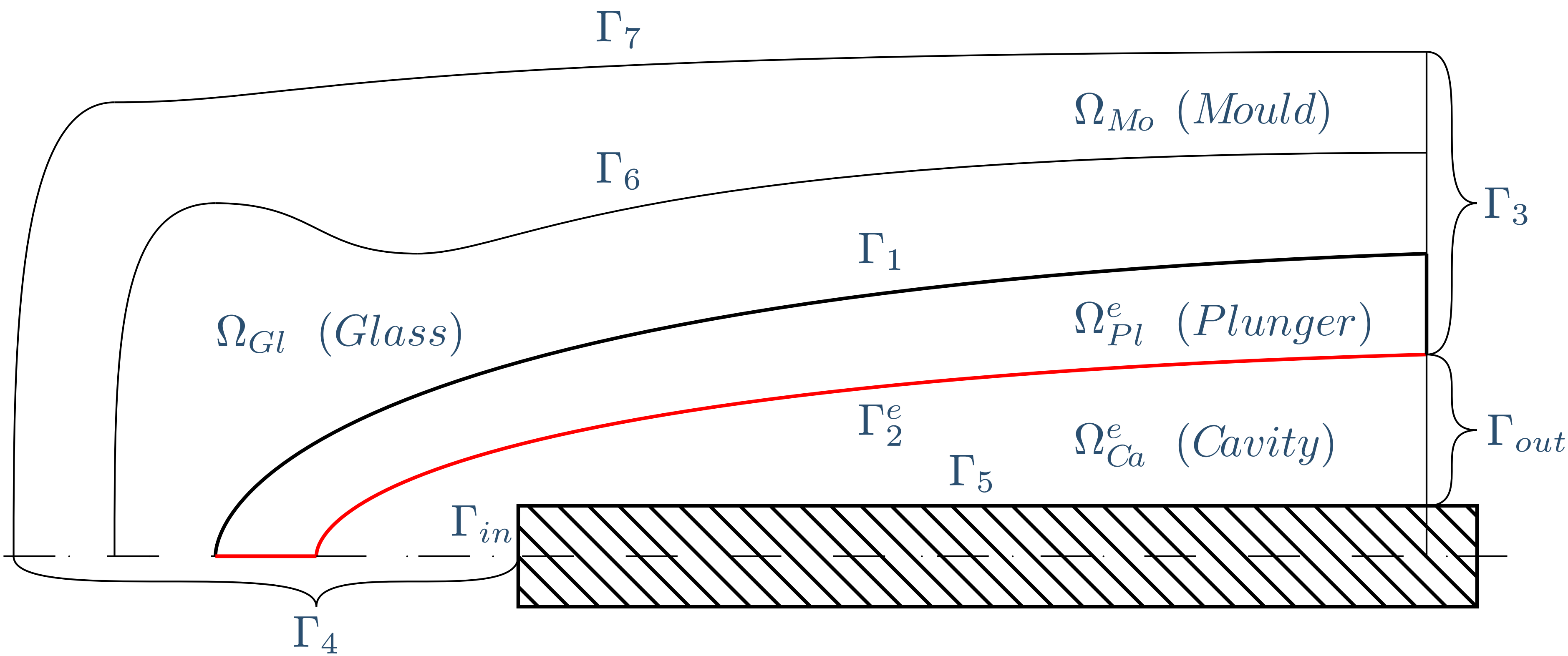

Fig 2

We denote $\Omega^S = \Omega^e_{Pl} \cup \Omega_{Gl} \cup \Omega^e_{Ca} \cup \Omega_{Mo}$.
In the three dimensional region G^e_{Ca}, which is created by rotation of Ω^e_{Ca} around x axis, we assume an axisymmetric incompressible potential flow of water, which is axisymmetric with x axis.

Shape optimization of the cooling plunger cavity

The potential Φ is given as a solution of Neumann problem

$$\Delta\Phi = 0 \qquad \text{in} \quad G^e_{Ca} \,, \tag{2}$$

$$\frac{\partial\Phi}{\partial n} = g \qquad \text{on} \quad \partial G^e_{Ca} \,, \tag{3}$$

where $g \in L^2(\partial G^e_{Ca})$, representing normal component of velocity at entrance to and exit of plunger cavity, is in the form

$$g = \begin{cases} 0 & \text{on} \quad \Gamma^{3D}_2 \cup \Gamma^{3D}_5 \,, \\ h^{in}_{velo} & \text{on} \quad \Gamma^{3D}_{in} \,, \\ h^{out}_{velo} & \text{on} \quad \Gamma^{3D}_{out} \,, \end{cases} \tag{4}$$

h^{in}_{velo} normal velocity at the entrance Γ^{3D}_{in}, $(h^{in}_{velo} < 0)$ and h^{out}_{velo} normal velocity at the exit Γ^{3D}_{out}. Further we assume

$$\int_{\Gamma^{3D}_{in} \cup \Gamma^{3D}_{out}} g \, dS = 0 \,. \tag{5}$$

Shape optimization of the cooling plunger cavity

The variational formulation for the potential function has form:
We look for the function $\Phi \in H^1(G^e_{Ca})$ such that

$$\int_{G^e_{Ca}} \frac{\partial \Phi}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dV = \int_{\Gamma^{3D}_{in} \cup \Gamma^{3D}_{out}} g \varphi \, dS \quad \forall \varphi \in H^1(G^e_{Ca}) \ . \tag{6}$$

Velocity field of flowing water $\boldsymbol{u} = (u_1, u_2, u_3)$ in cavity G^e_{Ca} is given as

$$\boldsymbol{u} = \text{grad}\Phi \ . \tag{7}$$

Theorem *(existence and uniqueness of velocity field)*
Under the assumption (5) there exists unique velocity field of the form (7) satisfying the estimate of Euclid norm in the form

$$|||\boldsymbol{u}|||_{L^2(G^e_{Ca})} \leq c \left(\|h^{in}_{velo}\|_{L^2(\Gamma^{3D}_{in})} + \|h^{out}_{velo}\|_{L^2(\Gamma^{3D}_{out})} \right) \ . \tag{8}$$

Proof See [1].

Shape optimization of the cooling plunger cavity

The energy equation for stationary flow $\boldsymbol{u}$ with steady temperature in three dimensions has the form

$$c_v \text{grad}\vartheta.\boldsymbol{u} - \frac{k}{\varrho}\Delta\vartheta = q \ . \qquad (9)$$

We put $\boldsymbol{u} = 0$ in G^e_{Pl}, G_{Gl} and G_{Mo} (regions created by rotation of Ω^e_{Pl}, Ω_{Gl} and Ω_{Mo} around x axis) because of there is no flowing liquid inside. Further we consider $q = 0$ in G^e_{Pl}, G^e_{Ca} and G_{Mo} (there is no heat sources inside). Denote $G = G^e_{Pl} \cup G_{Gl} \cup G^e_{Ca} \cup G_{Mo}$.
We divide the notion for searched function ϑ representing distribution of temperature in the system to the sum of four functions as

$$\vartheta = \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 \ ,$$

where

$$\vartheta_i = \begin{cases} \vartheta|_{G_i} & \text{in} \quad G_i \\ 0 & \text{in} \quad G \setminus G_i \end{cases} \qquad \text{for} \quad i = 0, 1, 2, 3 \ , \qquad (10)$$

($G_0 \equiv G^e_{Pl}$, $G_1 \equiv G_{Gl}$, $G_2 \equiv G^e_{Ca}$, $G_3 \equiv G_{Mo}$).
Further we denote by $\vartheta_i|_{\Gamma^{3D}_j}$ the trace of solution ϑ_i on the boundary Γ^{3D}_j for i, j if Γ^{3D}_j is a boundary of G_i.

Shape optimization of the cooling plunger cavity

We assume the following **boundary conditions:**
In the entrance the cooling water has constant temperature $15^o C$

$$\vartheta_2|_{\Gamma_{in}^{3D}} = 288 \quad \text{on } \Gamma_{in}^{3D} .$$

$$\vartheta_2|_{\Gamma_{out}^{3D}} = h_{out}^e \quad \text{on } \Gamma_{out}^{3D} .$$

Tube is isolated, thus

$$\frac{\partial \vartheta_2}{\partial n} = 0 \qquad \text{on } \Gamma_5^{3D} .$$

The boundary condition on Γ_3^{3D} is given as

$$\vartheta_i|_{\Gamma_3^{3D}} = h_3 \qquad \text{on } \Gamma_3^{3D} \quad i = 0,\, 1,\, 3 ,$$

Shape optimization of the cooling plunger cavity

Heat-transfer through Γ_2^{3D} (i. e. between plunger and water) is modeled as boundary condition for contact of two bodies

$$\left(-k_0\frac{\partial\vartheta_0}{\partial n}\right)^- = \left(-k_2\frac{\partial\vartheta_2}{\partial n}\right)^+ \qquad \text{on} \quad \Gamma_2^{3D}\ , \tag{11}$$

The boundary Γ_7^{3D} (i. e. between mould and environment) is modeled as boundary condition of the third kind for contact between body and environment

$$\left(-k_3\frac{\partial\vartheta_3}{\partial n}\right)^- = \alpha(\vartheta_3|_{\Gamma_7} - \vartheta_4) \qquad \text{on} \quad \Gamma_7^{3D}\ , \tag{12}$$

Shape optimization of the cooling plunger cavity

We use the transit condition for contact between two bodies, where one of them changes its state of matter because of the influence of solidification to describe heat-transfer through the boundary Γ_1^{3D} between glass piece and plunger. Thus

$$\left(k_1 \frac{\partial \vartheta_1}{\partial n}\right)^+ - \left(k_0 \frac{\partial \vartheta_0}{\partial n}\right)^- \quad = \quad \beta_1 \qquad \text{on} \quad \Gamma_1^{3D} \ , \tag{13}$$

Analogously we describe heat-transfer through the boundary Γ_6^{3D} between glass and mould.

$$\left(k_1 \frac{\partial \vartheta_1}{\partial n}\right)^+ - \left(k_3 \frac{\partial \vartheta_3}{\partial n}\right)^- \quad = \quad \beta_6 \qquad \text{on} \quad \Gamma_6^{3D} \ . \tag{14}$$

Shape optimization of the cooling plunger cavity

Due to rotational symmetry we transform the problem to cylindrical coordinates and use dimensional reduction to x, r coordinates.
By this way we obtain two dimensional velocity field of flowing water $\boldsymbol{w}^{\boldsymbol{e}} = (w_1, w_2)$ by relations

$$w_1 = u_1 \ , \tag{15}$$

$$w_2 = \sqrt{(u_2)^2 + (u_3)^2} \ , \tag{16}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is defined in (7).
Dimensional reduction leads to one more boundary condition in axis of the system Γ_4, which means that there is no heat flow in the normal direction to axis, thus

$$\frac{\partial \vartheta_i}{\partial n} = 0 \qquad \text{on} \quad \Gamma_4 \quad i = 0,\ 1,\ 2,\ 3 \ .$$

Shape optimization of the cooling plunger cavity

To define the state problem based on the variational formulation of energy equation in two dimensions we define the operators

$$\text{Energy}^{velo}_{\Omega^S}(\vartheta,\, \boldsymbol{w},\, \psi) = c_v \varrho_2 \int_{\Omega^e_{Ca}} \left(\frac{\partial \vartheta_2}{\partial x} w_1 + \frac{\partial \vartheta_2}{\partial r} w_2 \right) \psi r \, d\Omega \;, \tag{17}$$

$$\begin{aligned} \text{Energy}^{cond}_{\Omega^S}(\vartheta,\, \psi) = \; & k_0 \int_{\Omega^e_{Pl}} \left(\frac{\partial \vartheta_0}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_0}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega + \\ & + \; k_1 \int_{\Omega_{Gl}} \left(\frac{\partial \vartheta_1}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_1}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega + \\ & + \; k_2 \int_{\Omega^e_{Ca}} \left(\frac{\partial \vartheta_2}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_2}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega + \\ & + \; k_3 \int_{\Omega_{Mo}} \left(\frac{\partial \vartheta_3}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_3}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega \;, \end{aligned} \tag{18}$$

$$\text{Environment}_{\Omega^S}(\vartheta,\, \psi) = \int_{\Gamma_7} \alpha \vartheta_3 |_{\Gamma_7} \psi r \, d\Gamma \;, \tag{19}$$

(20)

Shape optimization of the cooling plunger cavity

$$\text{Source}_{\Omega^s}(\psi) = \varrho_1 \int_{\Omega_{Gl}} q\psi r \, d\Omega \ , \tag{21}$$

$$\text{Coeff}_{\Omega^s}(\psi) = \int_{\Gamma_1} \beta_1 \psi r \, d\Gamma + \int_{\Gamma_6} \beta_6 \psi r \, d\Gamma + \int_{\Gamma_7} \alpha \vartheta_4 \psi r \, d\Gamma \ , \tag{22}$$

Shape optimization of the cooling plunger cavity

We denote the weighted Sobolev space $H_r^1(\Omega_i)$ with norm

$$\|v\|_{1,r,\Omega_i} = \left(\int_{\Omega_i} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial r} \right)^2 + v^2 \right] r \, d\Omega \right)^{\frac{1}{2}} \quad i = 0, 1, 2, 3 \; , \qquad (23)$$

$(\Omega_0 \equiv \Omega_{Pl}^e,\, \Omega_1 \equiv \Omega_{Gl},\, \Omega_2 \equiv \Omega_{Ca}^e,\, \Omega_3 \equiv \Omega_{Mo})$.

Further we denote

$$\boldsymbol{H}(\Omega) = \{\vartheta;\; \vartheta \text{ defined in (10)},\; \vartheta_i \in H_r^1(\Omega_i) \text{ for any } i = 0, 1, 2, 3\} \; .$$

We define the norm in $\boldsymbol{H}(\Omega)$ as

$$\|\vartheta\|_{\boldsymbol{H}} = \left(\|\vartheta_0\|_{1,r,\Omega_0}^2 + \|\vartheta_1\|_{1,r,\Omega_1}^2 + \|\vartheta_2\|_{1,r,\Omega_2}^2 + \|\vartheta_3\|_{1,r,\Omega_3}^2 \right)^{\frac{1}{2}} \; . \qquad (24)$$

Theorem The set $\boldsymbol{H}(\Omega)$ with the norm (24) is a Hilbert space.

Shape optimization of the cooling plunger cavity

Denote the set

$$\Omega_H = \Omega^S \cup \Gamma_3 \cup \Gamma_{in} \cup \Gamma_{out}$$

and

$$^e\mathcal{H}^{2D} = \{ v \in C^\infty(\Omega_H);\ v|_{\Gamma_3 \cup \Gamma_{in} \cup \Gamma_{out}} = 0 \} .$$

Let $\boldsymbol{H}_0(\Omega)$ be closure of the set $^e\mathcal{H}^{2D}$ according to the norm $\boldsymbol{H}(\Omega)$.
We assume existence of the function $\vartheta^e_\Gamma \in \boldsymbol{H}(\Omega)$ such that

$$\vartheta^e_\Gamma|_{\Gamma_{in}} = 288 \quad \text{on} \quad \Gamma_{in}, \tag{25}$$

$$\vartheta^e_\Gamma|_{\Gamma_{out}} = h^e_{out} \quad \text{on} \quad \Gamma_{out}, \tag{26}$$

$$\vartheta^e_\Gamma|_{\Gamma_3} = h_3 \quad \text{on} \quad \Gamma_3, \tag{27}$$

Shape optimization of the cooling plunger cavity

The State Problem:
We look for the function $\vartheta \equiv \vartheta(F_2^e) \in \boldsymbol{H}(\Omega)$ such that

$$\text{Energy}_{\Omega^S}^{velo}(\vartheta, \boldsymbol{w}^e, \psi) + \text{Energy}_{\Omega^S}^{cond}(\vartheta, \psi) + \text{Environment}_{\Omega^S}(\vartheta, \psi) =$$

$$= \text{Source}_{\Omega^S}(\psi) \quad + \quad \text{Coeff}_{\Omega^S}(\psi) \quad \forall \psi \in \ \boldsymbol{H}_0(\Omega)\ , \tag{28}$$

$$\vartheta - \vartheta_\Gamma^e \quad \in \quad \boldsymbol{H}_0(\Omega)\ , \tag{29}$$

where $F_2^e \in U_{ad}^e$ and $\boldsymbol{w}^e$ is corresponding flow pattern given as gradient of solution (6).

Shape optimization of the cooling plunger cavity

The physical assumption of cooling:

A1 Average temperature of water coming into the plunger cavity is less than average temperature of leaving water.

Theorem The bilinear form (17) satisfies the following condition

$$\text{Energy}_{\Omega^S}^{velo}(\vartheta,\, \boldsymbol{w}^e,\, \vartheta) > 0 \tag{30}$$

for ϑ, $\boldsymbol{w}^e$ satisfying the physical assumption of cooling **A1.**

Theorem *(existence and uniqueness of solution of the state problem)*
The state problem (28), (29) has a unique solution $\vartheta(F_2^e)$ for each $F_2^e \in U_{ad}^e$ and associated flow pattern $\boldsymbol{w}^e$ received as gradient of the unique solution of (6) and

$$\|\vartheta(F_2^e)\|_{\boldsymbol{H}} \leq \frac{1}{\min(c_0,\, c_1,\, c_2,\, c_3)} \|F_\Omega\|_{\boldsymbol{H}^*} \,. \tag{31}$$

Proof See [1].

Shape optimization of the cooling plunger cavity

We will solve the **problem of the optimal design for plunger cavity shape:**
We define the **cost functional** as

$$\mathcal{J}^S(F_2^e) = \|\vartheta(F_2^e)|_{\Gamma_1} - T_{\Gamma_1}\|^2_{0,r,\Gamma_1} \,, \tag{32}$$

where $\vartheta(F_2^e)|_{\Gamma_1}$ is the trace of solution $\vartheta(F_2^e)$ of state problem (28), (29) in the region Ω^e_{Pl} on the boundary Γ_1, T_{Γ_1} is before chosen fixed constant corresponding to optimal surface plunger temperature. We look for **optimal design** $F_{Opt} \in U^e_{ad}$ such that

$$\mathcal{J}^S(F_{Opt}) \leq \mathcal{J}^S(F_2^e) \quad \forall\, F_2^e \in U^e_{ad} \,. \tag{33}$$

Theorem *(existence of solution of the problem of optimal design for plunger cavity shape)*
The optimal design problem (33) has at least one solution.

Proof See [1].

References:

[1] Salač, P.: *Optimal design of the cooling plunger cavity*, Appl. Math. (accepted to publish).

Thank you for your attention.