# Discontinuous Galerkin method for the solution of compressible flow in time-dependent domains and fluid-structure interaction

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Goal: to work out a sufficiently accurate, robust and theoretically based method for the numerical solution of compressible flow with a wide range of Mach numbers and Reynolds numbers

Difficulties:

nonlinear convection dominating over diffusion $\Longrightarrow$

- boundary layers, wakes for large Reynolds numbers
- shock waves, contact discontinuities for large Mach numbers
- instabilities caused by acoustic effects for low Mach numbers

One of promising, efficient methods for the solution of compressible flow is the discontinuous Galerkin finite element method (DGFEM) using piecewise polynomial approximation of a sought solution without any requirement on the continuity between neighbouring elements.

– Reed&Hill 1973, LeSaint&Raviart 1974, Johnson&Pitkäranta 1986

– Cockburn&Shu 1989, Bassi&Rebay, Baumann&Oden 1997, ... Hartmann, Houston, ... van der Vegt, ... M.F., Dolejší, Kučera

– theory for elliptic or parabolic problems: Arnold, Brezzi, Marini, et al, Schwab, Suli,..., Wheeler, Girault, Riviere, ...

– theory for nonstationary (nonlinear) convection-diffusion problems

Prague school:

M.F., Dolejší, Sobotíková, Kučera, Vlasák

Švadlenka, Hájek, Česenek, Hozman, Holík, Hasnedlová, Šebestová, Hozman, Kosík, Hadrava ...

Here:

- analysis of the DGFEM for the solution of a nonlinear nonstationary convection-diffusion equation (= a simple prototype of the compressible Navier-Stokes system)

- applications to the simulation of compressible flow

## Continuous model problem

Find $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

$$\text{a)}\ \frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u)) = g \quad \text{in } Q_T, \qquad (1)$$

$$\text{b)}\ u\big|_{\partial\Omega\times(0,T)} = u_D,$$

$$\text{c)}\ u(x,0) = u^0(x),\ \ x \in \Omega.$$

$\Omega \subset \boldsymbol{R}^d$, $d = 2, 3$ - a bounded polygonal (if $d = 2$) or polyhedral (if $d = 3$) domain with Lipschitz-continuous boundary $\partial\Omega$ and $T > 0$

$g : Q_T \to \boldsymbol{R}$, $u_D : \partial\Omega \times (0, T) \to \boldsymbol{R}$, $u^0 : \Omega \to \boldsymbol{R}$ - given functions,

$f_s \in C^1(\boldsymbol{R})$, $s = 1, \ldots, d$, - prescribed fluxes

$$\beta : \boldsymbol{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty,$$

$$|\beta(u_1) - \beta(u_2)| \le L|u_1 - u_2|, \quad \forall u_1, u_2 \in \boldsymbol{R}.$$

## DG space semidiscretization

Let $\mathcal{T}_h$ ($h > 0$) be a *partition* of the closure $\overline{\Omega}$ of the domain $\Omega$ into a finite number of closed triangles ($d = 2$) or tetrahedra ($d = 3$) $K$ with mutually disjoint interiors such that

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K. \tag{2}$$

We call $\mathcal{T}_h$ a *triangulation* of $\Omega$ and do not require the standard conforming properties from the finite element method.

$h_K = \operatorname{diam}(K)$, $\quad h = \max_{K \in \mathcal{T}_h} h_K$, $\quad \rho_K$ = radius of the largest ball inscribed into $K$

$K, K' \in \mathcal{T}_h$ - neighbours - they have a common face

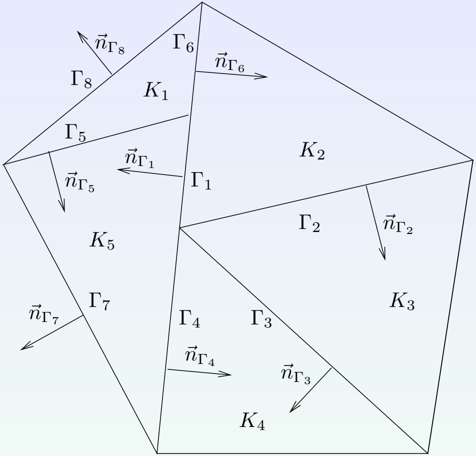
$\mathcal{F}_h$= the system of all faces of all elements $K \in \mathcal{T}_h$,
the set of all innner faces:

$$\mathcal{F}_h^I = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Omega\}, \tag{3}$$

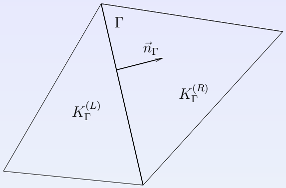
the set of all boundary faces:

$$\mathcal{F}_h^B = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega\}, \tag{4}$$

For each $\Gamma \in \mathcal{F}_h$ we define a *unit normal vector* $\boldsymbol{n}_\Gamma$.
For $\Gamma \subset \partial\Omega$ - $\boldsymbol{n}_\Gamma$ = unit outer normal to $\partial\Omega$.
$d(\Gamma)$ = diameter of $\Gamma \in \mathcal{F}_h$.




Elements with hanging nodes




Neighbouring elements

- For each face $\Gamma \in \mathcal{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$.
- $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$ and the inner normal to $\partial K_\Gamma^{(R)}$.
- If $\Gamma \in \mathcal{F}_h^B$, then $K_\Gamma^{(L)}$ will denote the element adjacent to $\Gamma$.

- Let $C_W > 0$ be a fixed constant. We set

$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2C_W} \quad \text{for } \Gamma \in \mathcal{F}_h^I, \tag{5}$$

$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}}}{C_W} \quad \text{for } \Gamma \in \mathcal{F}_h^B.$$

- Other possibility (if $\mathcal{T}_h$ is conforming):

$$h(\Gamma) = \frac{d(\Gamma)}{C_W} \quad \text{for } \Gamma \in \mathcal{F}_h. \tag{6}$$

## DG spaces:

- Broken Sobolev spaces:
  $H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K)\ \forall\, K \in \mathcal{T}_h\}$.
  - If $v \in H^1(\Omega, \mathcal{T}_h)$ and $\Gamma \in \mathcal{F}_h$, then
    $v_\Gamma^{(L)},\ v_\Gamma^{(R)}$ = the traces of $v$ on $\Gamma$ from the side of elements $K_\Gamma^{(L)},\ K_\Gamma^{(R)}$ adjacent to $\Gamma$
  - If $\Gamma \in \mathcal{F}_h^I$, then
    $\langle v \rangle_\Gamma = \frac{1}{2}\left(v_\Gamma^{(L)} + v_\Gamma^{(R)}\right)$, $[v]_\Gamma = v_\Gamma^{(L)} - v_\Gamma^{(R)}$.
- The approximate solution – sought in the space of discontinuous piecewise polynomial functions

$$S_h^p = \{v; v|_K \in P^p(K)\ \forall\, K \in \mathcal{T}_h\},$$

$p > 0$ – integer, $P^p(K)$ – the space of all polynomials on $K$ of degree at most $p$.

## Derivation of the discrete problem

Assume that $u$ – sufficiently regular exact solution
– multiply the PDE by any $\varphi \in H^2(\Omega, \mathcal{T}_h)$
– integrate over $K \in \mathcal{T}_h$
– apply Green's theorem
– sum over all $K \in \mathcal{T}_h$
– add some terms mutually vanishing

After some manipulation we obtain the identity

$$
\begin{aligned}
&\int_\Omega \frac{\partial u}{\partial t}\,\varphi\,\mathrm{d}x \qquad (7)\\
&+\sum_{K\in\mathcal{T}_h}\sum_{\substack{\Gamma\in\mathcal{F}_h\\ \Gamma\subset\partial K}}\int_\Gamma \sum_{s=1}^{d} f_s(u)\,(n_{\partial K})_s\,\varphi|_\Gamma\,\mathrm{d}S\\
&\quad-\sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^{d} f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x\\
&+\sum_{K\in\mathcal{T}_h}\int_K \beta(u)\,\nabla u\cdot\nabla\varphi\,\mathrm{d}x\\
&\quad-\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \langle\beta(u)\nabla u\rangle\cdot\boldsymbol{n}_\Gamma[\varphi]\,\mathrm{d}S\\
&\quad-\sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma \beta(u)\,\nabla u\cdot\boldsymbol{n}_\Gamma\,\varphi\,\mathrm{d}S=\int_\Omega g\,\varphi\,\mathrm{d}x.
\end{aligned}
$$

# Forms

## Forms

For $u, v, \varphi \in H^2(\Omega, \mathcal{T}_h)$, we define the folowing forms:

- Diffusion form

$$a_h(v,u,\varphi) = \sum_{K\in\mathcal{T}_h} \int_K \beta(v)\nabla u \cdot \nabla \varphi \,\mathrm{d}x \tag{8}$$
$$- \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma (\langle \beta(v)\nabla u\rangle \cdot \boldsymbol{n}_\Gamma [\varphi] + \theta\langle\beta(v)\nabla\varphi\rangle \cdot \boldsymbol{n}_\Gamma\,[u])\,\mathrm{d}S$$
$$- \sum_{\Gamma\in\mathcal{F}_h^B} \int_\Gamma (\beta(v)\nabla u \cdot \boldsymbol{n}_\Gamma\,\varphi$$
$$+\theta\,\beta(v)\nabla\varphi \cdot \boldsymbol{n}_\Gamma\,u - \theta\beta(v)\nabla\varphi \cdot \boldsymbol{n}_\Gamma\,u_D)\,\mathrm{d}S$$

$\theta = -1$, or $\theta = 0$ or $\theta = 1$ – the nonsymmetric (NIPG) or incomplete (IIPG) or symmetric (SIPG) variants of the approximation of the diffusion terms, respectively.

- Interior and boundary penalty

$$J_h(u,\varphi) = \sum_{\Gamma\in\mathcal{F}_h^I} h(\Gamma)^{-1}\int_\Gamma [u]\,[\varphi]\,\mathrm{d}S$$

$$+\sum_{\Gamma\in\mathcal{F}_h^B} h(\Gamma)^{-1}\int_\Gamma u\,\varphi\,\mathrm{d}S$$

$$A_h = a_h + \beta_0 J_h, \tag{9}$$

- Right-hand side form

$$\ell_h(\varphi) = (g,\varphi) + \beta_0 \sum_{\Gamma\in\mathcal{F}_h^B} h(\Gamma)^{-1}\int_\Gamma u_D\,\varphi\,\mathrm{d}S \tag{10}$$

- Convection form

$$b_h(u,\varphi) = -\sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^{2} f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \qquad (11)$$
$$+\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma H\left(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \boldsymbol{n}_\Gamma\right)[\varphi]\,\mathrm{d}S$$
$$+\sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma H\left(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \boldsymbol{n}_\Gamma\right)\varphi\,\mathrm{d}S$$

- $H$ – numerical flux with the following properties:
  - $H(u,v,\boldsymbol{n})$ is defined in $\boldsymbol{R}^2\times B_1$, where $B_1=\{\boldsymbol{n}\in\boldsymbol{R}^2; |\boldsymbol{n}|=1\}$, and is Lipschitz-continuous with respect to $u$, $v$.
  - $H(u,v,\boldsymbol{n})$ is consistent: $H(u,u,\boldsymbol{n})=\sum_{s=1}^2 f_s(u)\,n_s, u\in\boldsymbol{R}, \boldsymbol{n}=(n_1,n_2)\in B_1$.
  - $H(u,v,\boldsymbol{n})$ is conservative: $H(u,v,\boldsymbol{n})=-H(v,u,-\boldsymbol{n}),\quad u,\ v\in\boldsymbol{R},\ \ \boldsymbol{n}\in B_1$.

The exact sufficiently regular solution $u$ satisfies the identity

$$\left(\frac{\partial u(t)}{\partial t}, \varphi_h\right) + b_h(u(t), \varphi_h) + a_h(u(t), u(t), \varphi_h)$$
$$+\beta_0 J_h(u(t), \varphi_h) = \ell_h(\varphi_h)(t) \quad \text{for all } \varphi_h \in S_h^p \text{ and for a.e. } t \in (0, T).$$

$(\cdot, \cdot)$ – $L^2(\Omega)$-scalar product

**Discrete problem**

We say that $u_h$ is a DG approximate solution of the convection-diffusion problem (1), if

$$\text{a)} \qquad u_h \in C^1([0, T]; S_h^p), \tag{12}$$

$$\text{b)} \qquad \left(\frac{\partial u_h(t)}{\partial t}, \varphi_h\right) + a_h(u_h(t), u_h(t), \varphi_h) + b_h(u_h(t), \varphi_h) \tag{13}$$
$$+\beta_0 J_h(u_h(t), \varphi_h) = \ell_h(\varphi_h)(t) \quad \forall \varphi_h \in S_h^p, \ \forall t \in (0, T),$$

$$\text{c)} \qquad u_h(0) = u_h^0 = S_h^p\text{-approximation of } u^0.$$

Remark: Integrals are evaluated with the aid of *numerical integration*.

The discrete problem is equivalent to a large system of nonlinear ordinary differential equations.
In practical computations: suitable *time discretization* is applied, e.g.
– Euler forward or backward scheme, Crank-Nicolson
– Runge–Kutta methods,
The forward Euler and Runge-Kutta schemes are *conditionally stable* – time step is strongly restricted by the *CFL-stability condition*.
Suitable: *semi-implicit scheme* - leads to a linear algebraic system on each time level
– discontinuous Galerkin time discretization

# Space-time DGM

M.F. & J. Česenek

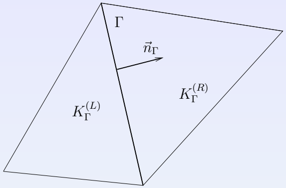
## Space-time DGM

## Space-time discretization

- Partition in the time interval $[0, T]$: $0 = t_0 < \cdots < t_M = T$ denote $I_m = (t_{m-1}, t_m)$, $\tau_m = t_m - t_{m-1}$, $\tau = \max_{m=1,\dots,M} \tau_m$.
- For $\varphi$ defined in $\bigcup_{m=1}^{M} I_m$ we put
  $\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t\to t_m\pm} \varphi(t)$ (one-sided limits at time $t_m$)
  $\{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-)$ (jump).
- For each $I_m$ consider a partition $\mathcal{T}_{h,m}$ of the closure $\overline{\Omega}$ of the domain $\Omega$ into a finite number of closed triangles with mutually disjoint interiors.

  The partitions $\mathcal{T}_{h,m}$ are in general different for different $m$.
- $\mathcal{F}_{h,m}$ – the system of all faces of all elements $K \in \mathcal{T}_{h,m}$
  $\mathcal{F}_{h,m}^{I}$ – the set of all inner faces
  $\mathcal{F}_{h,m}^{B}$ – the set of all boundary faces

- Each $\Gamma \in \mathcal{F}_{h,m}$ associated with a unit normal vector $\boldsymbol{n}_\Gamma$, which has the same orientation as the outer normal to $\partial\Omega$ for $\Gamma \in \mathcal{F}_{h,m}^B$
- $h_K = \mathrm{diam}(K)$ for $K \in \mathcal{T}_{h,m}$,
  $h_m = \max_{K \in \mathcal{T}_{h,m}} h_K, \; h = \max_{m=1,\dots,M} h_m$
  $\rho_K$ – the radius of the largest circle inscribed into $K$.




Neighbouring elements

- For each face $\Gamma \in \mathcal{F}_{h,m}^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_{h,m}$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$.
- $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$ and the inner normal to $\partial K_\Gamma^{(R)}$.
- If $\Gamma \in \mathcal{F}_{h,m}^B$, then $K_\Gamma^{(L)}$ will denote the element adjacent to $\Gamma$.

- Let $C_W > 0$ be a fixed constant. We set

$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2C_W} \quad \text{for } \Gamma \in \mathcal{F}_{h,m}^I, \tag{14}$$

$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}}}{C_W} \quad \text{for } \Gamma \in \mathcal{F}_{h,m}^B,$$

- or

$$h(\Gamma) = \frac{d(\Gamma)}{C_W} \quad \text{for } \Gamma \in \mathcal{F}_{h,m}. \tag{15}$$

# DG spaces:

- Broken Sobolev spaces:
  $H^k(\Omega, \mathcal{T}_{h,m}) = \{v; v|_K \in H^k(K)\ \forall K \in \mathcal{T}_{h,m}\}$.
  - If $v \in H^1(\Omega, \mathcal{T}_{h,m})$ and $\Gamma \in \mathcal{F}_{h,m}$, then $v_\Gamma^{(L)},\ v_\Gamma^{(R)}$ = the traces of $v$ on $\Gamma$ from the side of elements $K_\Gamma^{(L)},\ K_\Gamma^{(R)}$ adjacent to $\Gamma$
  - If $\Gamma \in \mathcal{F}_{h,m}^I$, then
    $\langle v \rangle_\Gamma = \frac{1}{2}\left(v_\Gamma^{(L)} + v_\Gamma^{(R)}\right),\ [v]_\Gamma = v_\Gamma^{(L)} - v_\Gamma^{(R)}$.
- Discrete spaces
  - Let $p,\ q \geq 1$ be integers. For each $m = 1, \ldots, M$,

$$S_{h,m}^p = \left\{\varphi \in L^2(\Omega); \varphi|_K \in P^p(K)\ \forall K \in \mathcal{T}_{h,m}\right\}. \qquad (16)$$

  - The approximate solution is sought in the space

$$S_{h,\tau}^{p,q} = \Big\{\varphi \in L^2(Q_T); \varphi\Big|_{I_m} = \sum_{i=0}^{q} t^i\, \varphi_i \qquad (17)$$

$$\text{with } \varphi_i \in S_{h,m}^p,\ \ m = 1, \ldots, M\Big\}.$$

# Forms

## Forms

For $u, v, \varphi \in H^2(\Omega, \mathcal{T}_{h,m})$, we define the folowing forms:

- Diffusion form

$$a_{h,m}(v, u, \varphi) = \sum_{K \in \mathcal{T}_{h,m}} \int_K \beta(v)\nabla u \cdot \nabla \varphi \, \mathrm{d}x \tag{18}$$
$$- \sum_{\Gamma \in \mathcal{F}^I_{h,m}} \int_\Gamma (\langle \beta(v)\nabla u \rangle \cdot \boldsymbol{n}_\Gamma [\varphi] + \theta \langle \beta(v)\nabla \varphi \rangle \cdot \boldsymbol{n}_\Gamma \, [u]) \, \mathrm{d}S$$
$$- \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma (\beta(v)\nabla u \cdot \boldsymbol{n}_\Gamma \, \varphi$$
$$+ \theta \, \beta(v)\nabla \varphi \cdot \boldsymbol{n}_\Gamma \, u - \theta \beta(v)\nabla \varphi \cdot \boldsymbol{n}_\Gamma u_D) \, \mathrm{d}S$$

$\theta = -1$, or $\theta = 0$ or $\theta = 1$ – the symmetric (SIPG) or incomplete (IIPG) or nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively.

- Interior and boundary penalty

$$J_{h,m}(u,\varphi) = \sum_{\Gamma\in\mathcal{F}^I_{h,m}} h(\Gamma)^{-1}\int_\Gamma [u]\,[\varphi]\,\mathrm{d}S$$
$$+ \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1}\int_\Gamma u\,\varphi\,\mathrm{d}S$$
$$A_{h,m} = a_{h,m} + \beta_0 J_{h,m}, \tag{19}$$

- Right-hand side form

$$\ell_{h,m}(\varphi) = (g,\varphi) + \beta_0 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1}\int_\Gamma u_D\,\varphi\,\mathrm{d}S \tag{20}$$

- Convection form

$$b_{h,m}(u,\varphi) = -\sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{s=1}^{2} f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \qquad (21)$$

$$+\sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma H\left(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \boldsymbol{n}_\Gamma\right)[\varphi]\,\mathrm{d}S$$

$$+\sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma H\left(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \boldsymbol{n}_\Gamma\right)\varphi\,\mathrm{d}S$$

- $H$ – numerical flux with the following properties:
  - $H(u, v, \boldsymbol{n})$ is defined in $\boldsymbol{R}^2 \times B_1$, where $B_1 = \{\boldsymbol{n} \in \boldsymbol{R}^2; |\boldsymbol{n}| = 1\}$, and is Lipschitz-continuous with respect to $u$, $v$.
  - $H(u, v, \boldsymbol{n})$ is consistent: $H(u, u, \boldsymbol{n}) = \sum_{s=1}^{2} f_s(u)\, n_s,\, u \in \boldsymbol{R}, \boldsymbol{n} = (n_1, n_2) \in B_1$.
  - $H(u, v, \boldsymbol{n})$ is conservative: $H(u, v, \boldsymbol{n}) = -H(v, u, -\boldsymbol{n}), \quad u,\ v \in \boldsymbol{R},\ \ \boldsymbol{n} \in B_1$.

- $(\cdot, \cdot)$ – the scalar product in $L^2(\Omega)$,
- $\| \cdot \|$ – the norm in $L^2(\Omega)$.
- $\|\varphi\|_{DG,m} = \left( \sum_{K \in \mathcal{T}_{h,m}} |\varphi|^2_{H^1(K)} + J_{h,m}(\varphi, \varphi) \right)^{1/2}$ – norm in $H^1(\Omega, \mathcal{T}_{h,m})$

## Approximate solution:

notation: $U' = \partial U/\partial t, u' = \partial u/\partial t$.
$U \in S^{p,q}_{h,\tau}$ such that

$$\int_{I_m} \left((U', \varphi) + A_{h,m}(U, U, \varphi) + b_{h,m}(U, \varphi)\right) \mathrm{d}t \tag{22}$$
$$+ \left(\{U\}_{m-1}, \varphi^+_{m-1}\right)$$
$$= \int_{I_m} \ell_{h,m}(\varphi)\,\mathrm{d}t, \quad \forall\, \varphi \in S^{p,q}_{h,\tau}, \quad m = 1, \ldots, M,$$
$$U_0^- = L^2(\Omega) - \text{projection of } u^0 \text{ on } S^p_{h,1}.$$

The exact regular solution $u$ satisfies the identity

$$\int_{I_m} \left((u', \varphi) + A_{h,m}(u, u, \varphi) + b_{h,m}(u, \varphi)\right) \mathrm{d}t \tag{23}$$
$$+ \left(\{u\}_{m-1}, \varphi^+_{m-1}\right)$$
$$= \int_{I_m} \ell_{h,m}(\varphi)\,\mathrm{d}t \quad \forall\, \varphi \in S^{p,q}_{h,\tau}, \quad \text{with } u(0-) = u^0.$$

# Error analysis

## Error analysis

- The main goal: analysis of the estimation of the error $e = U - u$
- $\Pi_m$ – the $L^2(\Omega)$-projection on $S^p_{h,m}$.
- $S^{p,q}_{h,\tau}$-interpolation $\pi$ of functions $v \in H^1(0,T;L^2(\Omega))$:

$$\text{a)} \quad \pi\, v \in S^{p,q}_{h,\tau}, \quad \text{b)} \quad (\pi\, v)\,(t_m-) = \Pi_m\, v(t_m-), \tag{24}$$

$$\text{c)} \quad \int_{I_m} (\pi v - v,\, \varphi^*)\, \mathrm{d}t = 0 \quad \forall\, \varphi^* \in S^{p,q-1}_{h,\tau}, \quad \forall\, m = 1, \ldots, M.$$

- $e = U - u = \xi + \eta$,
  $\xi = U - \pi u \in S^{p,q}_{h,\tau}$ and $\eta \;=\; \pi u - u$

$\Longrightarrow$ for each $\varphi \in S_{h,\tau}^{p,q}$:

$$\int_{I_m} \left( (\xi', \varphi) + A_{h,m}(U, U, \varphi) - A_{h,m}(u, u, \varphi) \right) \mathrm{d}t \qquad (25)$$
$$+ \left( \{\xi_{m-1}\}, \varphi_{m-1}^{+} \right)$$
$$= \int_{I_m} \left( b_{h,m}(u, \varphi) - b_{h,m}(U, \varphi) \right) \mathrm{d}t$$
$$- \int_{I_m} (\eta', \varphi) \mathrm{d}t - \left( \{\eta\}_{m-1}, \varphi_{m-1}^{+} \right).$$

## Derivation of an abstract error estimate

- Consider a system of triangulations $\mathcal{T}_{h,m}$, $m = 1, \dots, M$, $h \in (0, h_0)$, shape regular and locally quasiuniform:

$$\frac{h_K}{\rho_K} \le C_R, \quad \forall K \in \mathcal{T}_{h,m}, \tag{26}$$

$$h_{K_\Gamma^{(L)}} \le C_Q \, h_{K_\Gamma^{(R)}}, \quad h_{K_\Gamma^{(R)}} \le C_Q \, h_{K_\Gamma^{(L)}} \quad \forall \Gamma \in \mathcal{F}_{h,m}^I \tag{27}$$

Important tools in the analysis:

- multiplicative trace inequality:

$$\|v\|^2_{L^2(\partial K)} \le C_M \left( \|v\|_{L^2(K)} \, |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \qquad v \in H^1(K), \tag{28}$$

- inverse inequality:

$$|v|_{H^1(K)} \le C_I h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K). \tag{29}$$

- consistency of the form $b_{h,m}$: for each $k > 0$ there exists a constant $C = C(k)$ such that

$$
\begin{aligned}
&|b_{h,m}(U,\varphi) - b_{h,m}(u,\varphi)| \qquad (30)\\
&\le \frac{\beta_0}{k}\|\varphi\|^2_{DG,m} + C(\|\xi\|^2 + \|\eta\|^2 + \sum_{K\in\mathcal{T}_{h,m}} h_K^2|\eta|^2_{H^1(K)}).
\end{aligned}
$$

- coercivity of the diffusion form: Let

$$
\begin{aligned}
C_W &> 0, \quad \text{for } \theta = -1 \ (NIPG), && (31)\\
C_W &\ge \left(\frac{4\beta_1}{\beta_0}\right)^2 C_{MI} \quad \text{for } \theta = 1 \ (SIPG), && (32)\\
C_W &\ge 2\left(\frac{2\beta_1}{\beta_0}\right)^2 C_{MI} \quad \text{for } \theta = 0 \ (IIPG), && (33)
\end{aligned}
$$

where $C_{MI} = C_M(C_I + 1)(C_Q + 1)$. Then

$$
A_{h,m}(U,\xi,\xi) = a_{h,m}(U,\xi,\xi) + \beta_0 J_{h,m}(\xi,\xi) \ge \frac{\beta_0}{2}\|\xi\|^2_{DG,m}. \qquad (34)
$$

Let us substitute $\varphi := \xi$ in (25). Then

$$\left\|\xi_m^-\right\|^2 - \left\|\xi_{m-1}^-\right\|^2 + \frac{\beta_0}{2}\int_{I_m} \|\xi\|_{\mathrm{DG},m}^2 \,\mathrm{d}t \tag{35}$$

$$\leq \quad C\int_{I_m} \|\xi\|^2 \,\mathrm{d}t + 4\left\|\eta_{m-1}^-\right\|^2 + C\int_{I_m} R_m(\eta)\,\mathrm{d}t,$$

where

$$R_m(\eta) = \|\eta\|_{DG,m}^2 + \|\eta\|^2 + \sum_{K\in\mathcal{T}_{h,m}} \left(h_K^2|\eta|_{H^1(K)}^2 + h_K^2|\eta|_{H^2(K)}^2\right). \tag{36}$$

Necessary to estimate $\int_{I_m} \|\xi\|^2 \,\mathrm{d}t$

**Derivation of the estimate of $\int_{I_m} \|\xi\|^2 \, \mathrm{d}t$** – rather technical

(I) The case $\beta(u) = \mathrm{const} \; > 0$ analyzed by M.F., Kučera, Najzar and Prokopová in Numer. Math. 2011, using the approach based on the application of the so-called Gauss-Radau quadrature and interpolation.

(II) However, in the case of nonlinear diffusion, this technique is not applicable.

We use here the concept of the discrete characteristic functions to the function $\xi$ at points $y \in I_m$: $\tilde{\xi}_y \in S^{p,q}_{h,\tau}$,

$$\int_{I_m} (\tilde{\xi}_y, \varphi)\mathrm{d}t = \int_{t_{m-1}}^{y} (\xi, \varphi)\mathrm{d}t, \quad \forall \varphi \in S^{p,q-1}_{h,\tau} \quad \tilde{\xi}_y(t^+_{m-1}) = \xi(t^+_{m-1}).$$

The detailed analysis yields the estimate

$$\int_{I_m} \|\xi\|^2 \, \mathrm{d}t \leq C\, \tau_m \left( \left\|\xi^-_{m-1}\right\|^2 + \left\|\eta^-_{m-1}\right\|^2 + \int_{I_m} R_m(\eta) \, \mathrm{d}t \right).$$

The derived estimates and the discrete Gronwall lemma yield the abstract error estimate:

**Theorem 1** There exists a constants $C > 0$ such that the error $e = U - u$ satisfies the estimate

$$\|e_m^-\|^2 + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|e\|_{DG,j}^2\,\mathrm{d}t \tag{37}$$

$$\leq C\left(\sum_{j=1}^{m}\|\eta_j^-\|^2 + \sum_{j=1}^{m}\int_{I_j}R_j(\eta)\,\mathrm{d}t\right)$$

$$+2\|\eta_m^-\|^2 + 2\beta_0\sum_{j=1}^{m}\int_{I_j}\|\eta\|_{DG,j}^2\,\mathrm{d}t, \qquad m = 1, \dots, M.$$

# Error estimation in terms of $h$ and $\tau$

- the abstract error estimate
- estimation of terms containing $\eta$
- the assumptions on the regularity of the exact solution

$$u \in H^{q+1}\big(0, T; H^1(\Omega)\big) \cap C([0, T]; H^{p+1}(\Omega)), \quad (38)$$

- the assumptions on the properties of the meshes: shape regularity, quasiuniformity and

$$\tau_m \geq C h_m^2, \quad m = 1, \ldots, M. \quad (39)$$

- approximation properties of operators $\Pi_m, \ \pi$

If all meshes $\mathcal{T}_{h,m}$ are identical, then condition (39) can be omitted.
$\Longrightarrow$ error estimates in terms of $h$ and $\tau$:

**Theorem 2** There exists a constant $C > 0$ such that

$$\|e_m^-\|^2 + \frac{\varepsilon}{2} \sum_{j=1}^{m} \int_{I_m} \|e\|_{DG,j}^2 \,\mathrm{d}t \qquad (40)$$
$$\leq \quad C \left( h^{2p} |u|^2_{C([0,T];H^{p+1}(\Omega))} + \tau^{2q+\alpha} |u|_{H^{q+1}(0,T;H^1(\Omega))} \right).$$

Here $\alpha = 2$, if $u_D$ is a polynomial of degree $\leq q$ in $t$, otherwise $\alpha = 0$.

Further goals:

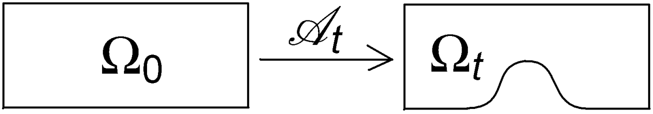
– derivation of optimal error estimates,

– demonstration of results by numerical experiments

# Compressible flow in a time-dependent domain - ALE method

Flow in a bounded time-dependent domain $\Omega_t \subset \mathbf{R}^2$, $t \in [0, T]$ - formulated with the aid of the ALE method, based on the ALE one-to-one regular mapping

$$\mathcal{A}_t : \overline{\Omega}_0 \to \overline{\Omega}_t, \text{ i.e. } \mathcal{A}_t : X \in \overline{\Omega}_0 \mapsto x = x(X,t) \in \overline{\Omega}_t.$$




Domain velocity:

$$\tilde{z}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X), t \in [0,T], X \in \Omega_0, \qquad (41)$$

$$z(x,t) = \tilde{z}(\mathcal{A}_t^{-1}(x),t), \; t \in [0,T], \; x \in \overline{\Omega}_t$$

$$(z|_{\Gamma_{W_t}} = z_D)$$

# Domain velocity, ALE derivative

ALE derivative of a function $f = f(x,t)$ defined for $x \in \Omega_t,\ t \in [0,T]$:

$$\frac{D^A}{Dt} f(x,t) = \frac{\partial \tilde{f}}{\partial t}(X,t)|_{X=\mathcal{A}_t^{-1}(x)}, \qquad (42)$$

where

$$\tilde{f}(X,t) = f(\mathcal{A}_t(X),t),\ X \in \Omega_0.$$

It is possible to show that

$$\frac{D^A f}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \mathrm{grad}\, f = \frac{\partial f}{\partial t} + \mathrm{div}(\boldsymbol{z} f) - f\, \mathrm{div} \boldsymbol{z}. \qquad (43)$$

$\Longrightarrow$ ALE formulation of the system describing compressible flow consisting of the continuity equation, the Navier-Stokes equations, the energy equation:

# ALE form of the governing equations

$$\frac{D^A \boldsymbol{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{g}_s(\boldsymbol{w})}{\partial x_s} + \boldsymbol{w}\,\mathrm{div}\boldsymbol{z} = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s}, \qquad (44)$$

where

$$\boldsymbol{w} = (w_1, \ldots, w_4)^T = (\rho, \rho v_1, \rho v_2, E)^T \in \boldsymbol{R}^4,$$
$$\boldsymbol{g}_s(\boldsymbol{w}) = \boldsymbol{f}_s(\boldsymbol{w}) - z_s \boldsymbol{w},$$
$$\boldsymbol{f}_s(\boldsymbol{w}) = (\rho v_s, \rho v_1 v_s + \delta_{1s}\, p, \rho v_2 v_s + \delta_{2s}\, p, (E+p) v_s)^T,$$
$$\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w}) = \left(0, \tau_{s1}^V, \tau_{s2}^V, \tau_{s1}^V\, v_1 + \tau_{s2}^V\, v_2 + k \partial\theta/\partial x_s\right)^T,$$
$$\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w}) = \sum_{k=1}^{2} \boldsymbol{K}_{sk}(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x_k},$$
$$\tau_{ij}^V = \lambda\, \mathrm{div} \boldsymbol{v}\, \delta_{ij} + 2\mu\, d_{ij}(\boldsymbol{v}),\ \ d_{ij}(\boldsymbol{v}) = \left(\partial v_i/\partial x_j + \partial v_j/\partial x_i\right)/2$$

## Thermodynamical relations

$$p = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2), \quad \theta = \left(E/\rho - |\boldsymbol{v}|^2/2\right)/c_v.$$

Notation: $\rho$ - density,
$p$ - pressure,
$E$ - total energy,
$\boldsymbol{v} = (v_1, v_2)$ - velocity,
$\theta$ - absolute temperature,
$\gamma > 1$ - Poisson adiabatic constant,
$c_v > 0$ - specific heat at constant volume,
$\mu > 0, \lambda = -2\mu/3$ - viscosity coefficients,
$k > 0$ - heat conduction

- Initial condition:

$$\boldsymbol{w}(\boldsymbol{x},0) = \boldsymbol{w}^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_0$$

- Boundary conditions: $\partial\Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_{W_t}$

Inlet $\Gamma_I$ :

$$\rho|_{\Gamma_I \times (0,T)} = \rho_D,$$

$$\boldsymbol{v}|_{\Gamma_I \times (0,T)} = \boldsymbol{v}_D = (v_{D1}, v_{D2})^T,$$

$$\sum_{j=1}^{2}\left(\sum_{i=1}^{2} \tau_{ij}^V n_i\right) v_j + k\frac{\partial\theta}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma_I \times (0,T);$$

Wall $\Gamma_{W_t}$ :

$$\boldsymbol{v}_{\Gamma_{W_t}} = \boldsymbol{z}, \quad \frac{\partial\theta}{\partial \boldsymbol{n}} = 0;$$

Outlet $\Gamma_O$ :

$$\sum_{i=1}^{2} \tau_{ij}^V n_i = 0, \quad \frac{\partial\theta}{\partial \boldsymbol{n}} = 0 \; j = 1,2;$$

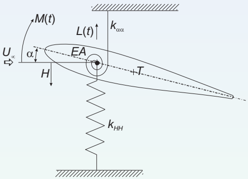
# Flow induced airfoil vibrations

## Flow induced airfoil vibrations

Flow induced vibrations of an elastically supported airfoil with two degrees of freedom:

– the vertical displacement $H$,

– the angle $\alpha$ of rotation around an elastic axis $EA$




The elastic support of the airfoil on translational and rotational springs

# Description of the airfoil motion

## Description of the airfoil motion

$$m\ddot{H} + k_{HH}H + S_\alpha\,\ddot{\alpha}\,\cos\alpha - S_\alpha\dot{\alpha}^2\sin\alpha + d_{HH}\dot{H} = -L(t), \qquad (45)$$
$$S_\alpha\ddot{H}\cos\alpha + I_\alpha\ddot{\alpha} + k_{\alpha\alpha}\alpha + d_{\alpha\alpha}\dot{\alpha} = M(t)$$

Initial conditions: $H(0), \alpha(0),\ \dot{H}(0), \dot{\alpha}(0)$

Physical data: $m, S_\alpha, I_\alpha, k_{HH}, k_{\alpha\alpha}, d_{HH}, d_{\alpha\alpha}$:

# Coupling of flow and structural problems

Coupling of flow and structural problems via the definition of

- $L$ - aerodynamic lift force,
- $M$ - aerodynamic torsional moment:

$$L = -\ell \int_{\Gamma_{Wt}} \sum_{j=1}^{2} \tau_{2j} n_j dS, \quad M = \ell \int_{\Gamma_{Wt}} \sum_{i,j=1}^{2} \tau_{ij} n_j r_i^{\mathrm{ort}} dS, \tag{46}$$

$$\tau_{ij} = -p\delta_{ij} + \tau_{ij}^{V}, \quad r_1^{\mathrm{ort}} = -(x_2 - x_{EA2}), \; r_2^{\mathrm{ort}} = x_1 - x_{EA1},$$

$\ell$ – airfoil depth

# Discrete problem

## Discretization of the flow problem

- construct a time partition $0 = t_0 < t_1 < t_2 \ldots$,
- the domain $\Omega_t$ is approximated by a polygonal domain $\Omega_h(t)$,
- triangulation $\mathcal{T}_h(t)$ in $\Omega_h(t)$.

# Forms in the discrete problems

**Forms in the discrete problem** - depend on time

Convection form (uses the relation $\boldsymbol{f}_s(\mathbf{w}) = \mathbf{A}_s(\mathbf{w})\mathbf{w}$ and the Vijayasundaram numerical flux)

$$b_h(\bar{\mathbf{w}}_h, \mathbf{w}_h, \mathbf{\Phi}_h, t) = -\sum_{K \in \mathcal{T}_h(t)} \int_K \sum_{s=1}^{2} (\mathbf{A}_s(\bar{\mathbf{w}}_h) - z_s(t)\mathbf{I})\, \mathbf{w}_h \cdot \frac{\partial \mathbf{\Phi}_h}{\partial x_s}\, \mathrm{d}\boldsymbol{x}$$

$$+ \sum_{\Gamma \in \mathcal{F}_h(t)^I} \int_\Gamma \left(\mathbf{P}^+\left(\langle \bar{\mathbf{w}}_h \rangle_\Gamma, \boldsymbol{n}_\Gamma\right) \mathbf{w}_h|_\Gamma + \mathbf{P}^-\left(\langle \bar{\mathbf{w}}_h \rangle_\Gamma, \boldsymbol{n}_\Gamma\right) \mathbf{w}_h|_\Gamma\right) \cdot [\mathbf{\Phi}_h]_\Gamma\, \mathrm{d}S$$

$$+ \sum_{\Gamma \in \mathcal{F}_h(t)^B} \int_\Gamma \left(\mathbf{P}^+\left(\langle \bar{\mathbf{w}}_h \rangle_\Gamma, \boldsymbol{n}_\Gamma\right) \mathbf{w}_h|_\Gamma + \mathbf{P}^-\left(\langle \bar{\mathbf{w}}_h \rangle_\Gamma, \boldsymbol{n}_\Gamma\right) \mathbf{w}_h|_\Gamma\right) \cdot \mathbf{\Phi}_h|_\Gamma\, \mathrm{d}S$$

## Viscosity form (IIPG)

$$a_h(\bar{\mathbf{w}}_h, \mathbf{w}_h, \mathbf{\Phi}_h, t) = \sum_{K \in cT_h(t)} \int_K \sum_{s=1}^{2} \sum_{k=1}^{2} \mathbf{K}_{s,k}(\bar{\mathbf{w}}_h) \, \frac{\partial \mathbf{w}_h}{\partial x_k} \cdot \frac{\partial \mathbf{\Phi}_h}{\partial x_s} \, \mathrm{d}\mathbf{x}$$

$$- \sum_{\Gamma \in \mathcal{F}_h(t)^I} \int_\Gamma \sum_{s=1}^{2} \left\langle \sum_{k=1}^{2} \mathbf{K}_{s,k}(\bar{\mathbf{w}}_h) \frac{\partial \mathbf{w}_h}{\partial x_k} \right\rangle_\Gamma (n_\Gamma)_s \cdot [\mathbf{\Phi}_h]_\Gamma \, \mathrm{d}S$$

$$- \sum_{\Gamma \in \mathcal{F}_h(t)^B} \int_\Gamma \sum_{s=1}^{2} \sum_{k=1}^{2} \mathbf{K}_{s,k}(\bar{\mathbf{w}}_h|_\Gamma) \left. \frac{\partial \mathbf{w}_h}{\partial x_k} \right|_\Gamma (n_\Gamma)_s \cdot \mathbf{\Phi}_h|_\Gamma \, \mathrm{d}S$$

## Reaction form

$$d_h(\mathbf{w}_h, \boldsymbol{\Phi}_h, t) = \sum_{K \in \mathcal{T}_h(t)} \int_K \operatorname{div} \boldsymbol{z}(t) \, (\mathbf{w}_h \cdot \boldsymbol{\Phi}_h) \, \mathrm{d}\boldsymbol{x}$$

## Interior and boundary penalty

$$J_h(\mathbf{w}_h, \boldsymbol{\Phi}_h, t) = \sum_{\Gamma \in \mathcal{F}_h(t)^I} h(\Gamma)^{-1} \int_\Gamma [\mathbf{w}_h]_\Gamma \cdot [\boldsymbol{\Phi}_h]_\Gamma \, \mathrm{d}S$$

$$+ \sum_{\Gamma \in \mathcal{F}_h(t)^B} h(\Gamma)^{-1} \int_\Gamma \mathbf{w}_h|_\Gamma \cdot \boldsymbol{\Phi}_h|_\Gamma \, \mathrm{d}S,$$

## Right-hand side form

$$\ell_h(\bar{\mathbf{w}}_h, \boldsymbol{\Phi}_h, t) = \mu \sum_{\Gamma \in \mathcal{F}_h(t)^B} h(\Gamma)^{-1} \int_\Gamma \mathbf{w}_B(t) \cdot \boldsymbol{\Phi}_h|_\Gamma \, \mathrm{d}S$$

For simple notation we define the forms

$$\begin{aligned} A_h(\bar{\mathbf{w}}_h, \mathbf{w}_h, \mathbf{\Phi}_h, t) &= a_h(\bar{\mathbf{w}}_h, \mathbf{w}_h, \mathbf{\Phi}_h, t) + b_h(\bar{\mathbf{w}}_h, \mathbf{w}_h, \mathbf{\Phi}_h, t) \\ &\quad + d_h(\mathbf{w}_h, \mathbf{\Phi}_h, t) + \mu J_h(\mathbf{w}_h, \mathbf{\Phi}_h, t). \end{aligned}$$

The approximate solution is sought in the space $\mathbf{S}_{h,\tau}^{p,q} = (S_{h,\tau}^{p,q})^4$, where

$$S_{h,\tau}^{p,q} = \left\{ \phi \; ; \; \phi|_{I_m} = \sum_{i=0}^{q} \zeta_i \phi_i, \ \text{kde } \phi_i \in S_h^p(t), \ \zeta_i \in P^q(t_{m-1}, t_m) \right\}.$$

# Approximate solution

## Linearized numerical scheme

Approximate solution: $\mathbf{w}_{h\tau}$ satisfying

$$
\begin{aligned}
&1)\ \mathbf{w}_{h\tau} \in \mathbf{S}^{p,q}_{h,\tau}, \qquad (47)\\
&2)\ \int_{I_m} \left( \left( \frac{D^{\mathcal{A}} \mathbf{w}_{h\tau}}{Dt}, \mathbf{\Phi}_{h\tau} \right)_t + A_h(\bar{\mathbf{w}}_{h\tau}, \mathbf{w}_{h\tau}, \mathbf{\Phi}_{h\tau}, t) \right) \mathrm{d}t \\
&\quad + \left( \{\mathbf{w}_{h\tau}\}_{m-1}, \mathbf{\Phi}_{h\tau}(t^+_{m-1}) \right) \\
&= \int_{I_m} \ell_h(\bar{\mathbf{w}}_{h\tau}, \mathbf{\Phi}_{h\tau}, t)\, \mathrm{d}t \quad \forall \mathbf{\Phi}_{h\tau} \in \mathbf{S}^{p,q}_{h,\tau}, \quad m = 1, \dots, M.
\end{aligned}
$$

$\bar{\mathbf{w}}_{h\tau}$ - prolongation from the time interval $I_{m-1}$ to $I_m$.

Spurious overshoots and undershoots may appear in the numerical solution at discontinuities or internal and boundary layers

To avoid them, we use

- local artificial viscosity (M.F., V. Kučera, 2007)
- based on the discontinuity indicator (M.F., V. Dolejší, C. Schwab, 2003)

Realization of the FSI carried by

- weak fluid-structure coupling or
- strong fluid-structure coupling

# Examples - flow-induced vibrations of the profile NACA0012

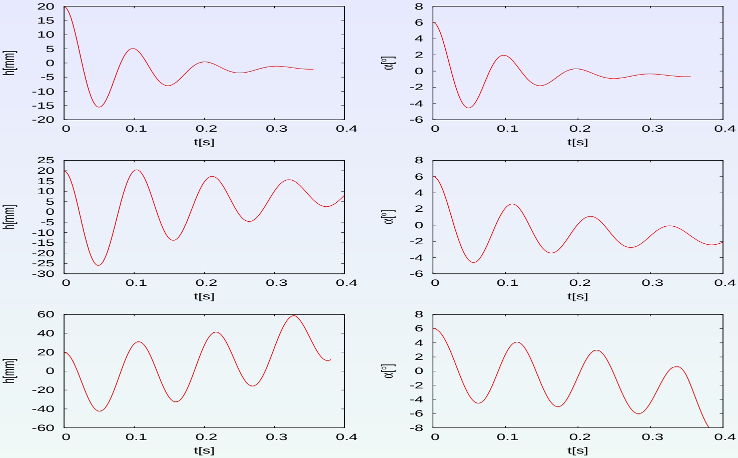
## Examples - flow-induced vibrations of the profile NACA0012

Initial conditions: $H(0) = 20$ mm, $\alpha(0) = 6^\circ, \dot{H}(0) = \dot{\alpha}(0) = 0$

### a) Subsonic flow

Far field velocities 30 and 35 m/s and Mach numbers 0.0882 and 0.1029, respectively: damped vibrations,

Far field velocity 40 m/s and Mach number 0.1176: flutter instability combined with a divergence instability - vibration amplitudes are increasing in time.




Displacement $H$ (left) and rotation angle $\alpha$ (right) of the airfoil in dependence on time for far-field velocity 30, 35 and 40 m/s

## b) Hypersonic flow

- Far field velocity 408 m/s, Mach number 1.2,
- Far field velocity 680 m/s, Mach number 2.0,
- Initial conditions: $H(0) = 20$ mm, $\alpha(0) = 6^\circ, \dot{H}(0) = \dot{\alpha}(0) = 0$,
- Bending and torsional stiffnesses - 1000times larger than for low Mach number flows
- $\Longrightarrow$ damped vibrations

# High-speed flow induced airfoil vibrations

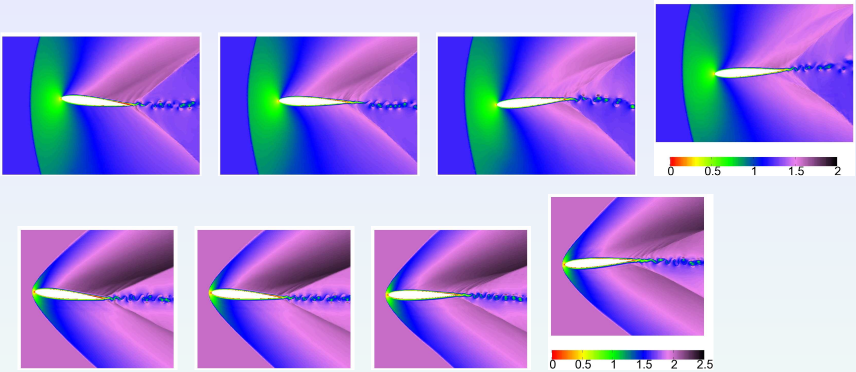



Figure: Distribution of the Mach number (Ma). Upper for far field Ma$= 1.2$ and $Re = 10^7$, lower for far field Ma$= 2.0$ and $Re = 10^7$ for different time instants

## Airfoil vibrations

# Future work

Future work:

- theory of continuous fluid-structure interaction problems
- further analysis of qualitative properties of the developed schemes
- coupling of compressible flow with nonlinear elastic materials
- including of turbulence models

Thank you for your attention


NÁVOD K POUŽITÍ
a) SPOJITÁ GALERKINOVA METODA
b) NESPOJITÁ GALERKINOVA METODA